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A new form of fuzzy compactness

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Abstract. The notion of βS^*- compactness is introduced by I. M.Hanafy in L-fuzzy topological spaces based on S^*-compactness.[βS^*- compactness in L-fuzzy topological spaces, J. Nonlinear Sci .Appl. 2(2009) ,no. 1, 27-37]. In this paper we introduced the notion of αS^*- compactness in L-fuzzy topological spaces based on α-compactness.We give some characterizations of αS^*-compactness and Some of its topological properties are discussed.

Key Words and Phrases: S^*- compactness, βS^*- compactness, αS^*- compactness

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1. Introduction

The concept of compactness is one of the most important concepts in general topology. The concept of compactness of a $[0,1]$-topological space was first introduced by Chang[4] in terms of open cover. Changs compactness has been greatly extended to the variable-basis case by Rodabaugh[16] and it may be regarded as a successful definition of compactness in poslat topology from the categorical point of view. Goguen[8] pointed out a deficiency in Changs compactness theory by showing that the Tychonoff Theorem is false. Since Changs compactness has some limitations, Gantner, Steinlage[6] and Warren introduced α-compactness , Lowen introduced fuzzy compactness[12], strong fuzzy compactness[13] and ultra-fuzzy compactness[14] and Wang[19] and Zhao[21] introduced N-compactness . Recently Shi[17] introduced $S^\star$ -compactness in L-fuzzy topological spaces. The notion of β-compactness is one of the good strong forms of compactness in topology. It was generalized and studied by many authors in fuzzy topological spaces.Also Shi[17]introduced a new notion of β-compactness in L-fuzzy topological spaces named β $S^\star$- compactness.

In this paper, we introduced a new notion of α compactness in L-fuzzy topological spaces named as $\alpha S^\star$-compactness.we introduced the notion of αS^*- compactness in L-fuzzy topological spaces based on α-compactness.We give some characterizations of αS^*-compactness and Some of its topological properties of αS^*-compactness are also discussed.

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2. Preliminaries

Throughout, this paper, $(L, \bigvee, \bigwedge, 0)$ is a completely distributive de Morgan algebra, and X a nonempty set. L^X is the set of all $L-fuzzy$ sets on X. An element a in L is called a prime element if $a \geq b \bigwedge c$ implies $a \geq b$ or $a \geq c$. a in L is called a co-prime element if a' is a prime element [7]. The set of non unite prime elements in L is denoted by $P(L)$. The set of nonzero co-prime elements in L is denoted by $M(L)$. The binary relation $\prec$ in L is defined as follows: for $a, b \in L, a \prec b$ iff for every subsets $D \subseteq L$, the relation $b \leq supD$ always implies the existence of $d \in D$ with $a \leq d$. In a completely distributive de Morgan algebra L ,each element b is a sup of $a \in L : a \prec b$. In the sense of [11], [19] $a \in L : a \prec b$ is the greatest minimal family of b, in symbol $\beta(b)$. Moreover for $b \in L$, define $\beta^*(b) = \beta(b) \cap M(L), \alpha(b) = \{a \in L : a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$. For $a \in L$ and $G \in L^X$, we denote $G^{(a)} = \{x \in X : G(x) \not\leq a\}$ and $G_{(a)} = \{x \in X : a \in \beta(G(x))\}$.[17], [18]

Definition 2.1. *An $L-$fuzzy set G in an $L-$ Fuzzy topological spaces (X, τ) is said to be*
(i) $\alpha-$open if $G \leq int\ cl\ int\ G$.
(ii)$\alpha-$closed if $G \geq cl\ int\ cl\ G$.
iii) preopen if $G \leq int\ cl\ G$.
(iv) preclosed if $G \geq cl\ int\ G$.
(v) $\beta-$open if $G \leq cl\ int\ cl\ G$.
(vi) $\beta-$closed if $G \geq int\ cl\ int\ G$.
(vii) regular open if $G = int\ cl\ G$.
(viii) regular closed if $G = cl\ int\ G$.
(ix)regular semiopen if there exist a regular open subset H of X such that $H \subseteq G \subseteq cl\ H$.
(x)regular semiclosed if there exist a regular closed subset H of X such that $H \supseteq G \supseteq int\ H$.

Definition 2.2. *A function $f : X \to Y$ is said to be fuzzy β-continuous [5] (resp. $M\beta$-continuous [9]) if the inverse image of every open (resp. β- open)L-fuzzy set in Y is β- open (resp. β- open) L-fuzzy set in X.*

Definition 2.3. *[17] Let (X, τ) be an L-fts, $a \in M(L)$ and $G \in L^X$. A subfamily ξ of L^X is called a β_a -cover of G if for any $x \in X$ with $a \notin \beta(G'(x))$,there exists an $A \in \xi$ such that $a \in \beta(A(x))$. A β_a- cover ξ of G is called an open (resp. regular open, preopen, etc.) β_a -cover of G if each member of ξ is open (resp. regular open, preopen, etc.) .*

It is obvious that ξ is a β_a- cover of G iff for any $x \in X$ it follows that $a \in \beta(G'(x) \vee \to A \in \xi \vee A(x))$.

Definition 2.4. *[17] Let (X, τ) be an L-fts, $a \in M(L)$ and $G \in L^X$. A subfamily ξ of L^X is called a Q_a -cover of G if for any $x \in X$ with $G(x) \not\leq a'$. it follows that $\to A \in \xi \vee A(x)) \geq a$. A Q_a- cover ξ of G is called an open (resp. regular open, preopen, etc.) Q_a -cover of G if each member of ξ is open (resp. regular open, preopen, etc.).*

Definition 2.5. *[17] Let (X, τ) be an L-fts, $a \in M(L)$ and $G \in L^X.G$ is called S^*-compact if for any $a \in M(L)$, each open β_a- cover of G has a finite subfamily F which is an open Q_a- cover of G.(X, τ) is said to be S^*-compact if $\rightarrow_1$ is S^*-compact.*

Definition 2.6. *[10] Let (X, τ) be an L - fts and $G \in L^X$ Then G is called βS^*-compact if for any $a \in M(L)$, every β - open β_a - cover of G has a finite subfamily F which is β-open Q_a-cover of G . (X, τ) is said to be βS^*-compact if X is βS^*-compact.*

3. Topological properties of αS^* - compactness

Definition 3.1. *A function $f : X \rightarrow Y$ is said to be fuzzy α-continuous (resp. $M\alpha$-continuous) if the inverse image of every open (resp. α- open)L-fuzzy set in Y is α-open (resp. α- open) L-fuzzy set in X.*

Definition 3.2. *Let (X, τ) be an L - fts and $G \in L^X$ Then G is called αS^*-compact if for any $a \in M(L)$, every α - open β_a - cover of G has a finite subfamily F which is α-open Q_a-cover of G . (X, τ) is said to be αS^*-compact if X is αS^*-compact.*

Theorem 3.3. *Let $f : X \rightarrow Y$ be fuzzy α-continuous surjection. If X is a α S^*-compact L - fts then Y is S^*-compact L - fts, where X and Y will be denote L - fts,s.*

Proof. For all $b \in M(L)$, let $(\nu_j : j \epsilon J)$ be a family of open L-fuzzy subsets of Y which is open β_b-cover of Y. Then $(f^{-1}(\nu_j) : j \in J)$ is a family of α-open L-fuzzy subsets of X which is α - open β_a - cover of X, for all $a \in M(L)$ where $f(a) = b$. From the α $S^\star$-compactness of X there exists a finite subset F of J which is α - openQ_a - cover of X.Hence $f(\rightarrow j \in F \vee f^{-1}(\nu_j)) = \rightarrow j \in F \vee f(\rightarrow j \in F \vee f^{-1}(\nu_j)) = \rightarrow j \in F \vee \nu_j$ and so is open Q_a - cover of X. which means that Y is $S^\star$-compact .

Theorem 3.4. *If $f : X \rightarrow Y$ is fuzzy open and fuzzy continuous function , then f is fuzzy $M\alpha$-continuous.*

Proof. Let H be a α - open L-fuzzy set in Y, then

$$H \leq int.cl.intH$$

so

$$f^{-1}(H) \leq f^{-1}(int.cl.intH) \tag{1}$$

since, f is fuzzy continuous,then

$$f^{-1}(int.cl.intH) = int(f^{-1}(cl.intH)) \tag{2}$$

also, clearly,

$$f^{-1}(cl.intH) \leq cl(f^{-1}(intH)) \tag{3}$$

since, f is fuzzy continuous, then

$$(f^{-1}(intH)) = intf^{-1}(H) \tag{4}$$

By 1,2, 3, 4 we get,

$$f^{-1}(H) \leq f^{-1}(int.cl.intH) = int(f^{-1}(cl.intH)) \leq int.cl.f^{-1}(intH) = int.cl.int(f^{-1}(H))$$

Hence ,

$$f^{-1}(H) \leq int.cl.int(f^{-1}(H))$$

Hence,f is fuzzy $M\alpha$-continuous.

Theorem 3.5. *Let $f : X \to Y$ be fuzzy $M\alpha$-continuous surjection. If X is a $M\alpha$-compact L - fts then Y is a $M\alpha$-compact L - fts.*

Proof.by using the definition of $M\alpha$ continuous function and 3.4, we get proof.

Theorem 3.6. *Let $f : X \to Y$ be a fuzzy $M\alpha$- open bijective function and Y is $M\alpha$-compact, then X is $M\alpha$-compact.*

Proof. For all $a \in M(L)$, let $(\nu_j : j \in J)$ be a family of α - open L-fuzzy subsets of X which is α - open β_a - cover of X. Then $(f(\nu_j) : j \in J)$ is a family of α - open L-fuzzy subsets of Y which is α - open β_b - cover of Y , for all $b \in M(L)$ where $f(a) = b$. From the α $S^\star$-compactness of Y there exists a finite subset F of J which is α - openQ_b - cover of Y . But $X = f^{-1}(Y) = f^{-1}f(\to j \in F \vee \nu_j) =\to j \in F \vee \nu_j$ which is α -open Q_a - cover of X and therefore X is α $S^\star$-compact.

Theorem 3.7. *Let (X, τ) be an L - fts. If G and H are α $S^\star$ -compact L-fuzzy subsets of X, then $G \vee H$ is also α $S^\star$ -compact L-fuzzy subsets of X.*

Proof. For any $a \in M(L)$, suppose that ξ is an α -open β_a -cover of $G \vee H$ Then by $(G \vee H)^{'}(x) \vee \to A \in \xi \vee A(x) = (G^{'}(x) \vee \to A \in \xi \vee A(x)) \wedge (H^{'}(x) \vee \to A \in \xi \vee A(x))$ we obtain that for any $x \in X$, $a \in \alpha(G^{'}(x) \vee \to A \in \xi \vee A(x))$ and $a \in \alpha(H^{'}(x) \vee \to A \in \xi \vee A(x))$ This shows that ξ is an α - open β_a - cover of G and H, we know that ξ has finite subfamily F_1 and F_2 such that F_1 and F_2 is a α- openQ_a - cover of G and H respectively.Hence for any $x \epsilon X$, $a \leqslant G^{'}(x) \vee \to A \epsilon F_1 \vee A(x)$ and $a \leqslant H^{'}(x) \vee \to A \epsilon F_2 \vee A(x)$ Take $W = F_1 \cup F_2$ is a finite subfamily of ξ and it satisfies the following condition, $a \leqslant G^{'}(x) \vee \to A \in W \vee A(x)$ and $a \leqslant H^{'}(x) \vee \to A \in W \vee A(x)$
hence, $a \leqslant (G \vee H)^{'}(x) \vee \longrightarrow A \in W \vee A(x)$. This shows that W is a α - open Q_a - cover of $G \vee H$, therefore $G \vee H$ is $\alpha S^\star$ -compact.

Theorem 3.8. *An L - fts (X, τ) is $\alpha S^\star$-compact if every α - closed fuzzy subset is α $S^\star$-compact relative to X.*

Proof. For any $a\epsilon M(L)$, suppose that $(\nu_j : j\epsilon J)$ be an α - open β_a - cover of X. Let $j_0\epsilon J$, then $\nu^{'}_{j_0}$ is α- closed and so by the hypothesis $\nu^{'}_{j_0}$ is α $S^{\star}$-compact.Now, $\xi = (\nu^{'}_{j_0} - j_0)$ is an α - open β_a - cover of X.Since $\nu^{'}_{j_0}$ is $\alpha S^{\star}$-compact there exists a finite subfamily ξ_0 of ξ such that ξ_0 is a α - open Q_a - cover of X. Hence X is a $\alpha S^{\star}$-compact.

Theorem 3.9. *Let $(X, w_L(\tau))$ be generated topology by (X, τ), Then χ_G is a α -open L-fuzzy set in $(X, w_L(\tau))$ if G is a α -open set in (X, τ).*

Proof. Since G is a α-open,by he definition α-open then $G \leq int.cl.intG$. Hence $\chi_G \leq \chi_{int.cl.intG} = int.cl.int\chi_G$ which implies that χ_G is a α - open L-fuzzy set in $(X, w_L(\tau))$.

Definition 3.10. *Let X be a set. A prefilterbase in X is a family $\Omega\epsilon L^X$ having the following two properties:*

(i) for every $G \in \Omega$, $G \neq \phi$

(ii) for every $G, H \in \Omega$ there is a $W \in \Omega$ such that $W \leqslant G \wedge H$.

Definition 3.11. *Let (X, τ) be an L - fts. A prefilterbase Ω on X is said to be α -converges to $a \in M(L)$ if for every α - open L-fuzzy set G containing 'a' there exists $H \in \Omega$ such that $H \leq intG$.*

Definition 3.12. *Let (X, τ) be an L - fts. A prefilterbase Ω on X is said to be α -accumulates at $a \in M(L)$ if for every α - open L-fuzzy set G containing 'a' and for every $H \in \Omega$. we have $H \wedge int.G \neq \phi$ such that $H \leq intG$.*

Theorem 3.13. *Let be a maximal prefilterbase in an L - fts (X, τ), then the following statements are equivalent:*
$(i)\Omega$ is α-accumulates at $a\epsilon M(L)$.
$(ii)\Omega$ is α-converges to $a\epsilon M(L)$.

Proof. $(i) \to (ii)$:To prove thatΩ is α-converges to $a \in M(L)$ Let G be a α-open L- fuzzy set in X such that $a \in G$. Since Ω is α-accumulates at $a \in M(L)$ then for every $H \in \Omega$, $H \wedge int.G \neq \phi$ Thus there exists a proper L- fuzzy subset $C \leq H$ such that $C \leq intG$.Since $C \neq \phi$, then C is a member of some prefilterbase in X. But Ω is maximal,then C is a member of Ω. Thus for every α- open L- fuzzy set G containing a there exists $H = C \in \Omega$ such that $H \leqslant intG$. Then Ω is α-converges to a. $(ii) \to (i)$: Let G be a α - open L- fuzzy set in X such that $a \in G$. Since is α-converges to a, then there exists $H \in \Omega$ such that $H \leqslant intG$ and thus $H \wedge intG$ is a member of some prefilterbase in X.But Ω is maximal, then $H \wedge intG \in \Omega$, So for every $H_j \in \Omega$,$H_j \wedge (H \wedge intG)$ contains a member of Ω, then $H_j \wedge G \neq \Omega$ for every $H_j \in \Omega$. Hence Ω is α -accumulates at a. ,

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