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Some New Generalizations and Extensions of Eneström-Kakeya Theorem

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Abstract. *If $P(z) = \sum_{j=0}^{n} a_j z^j, a_j \geq a_{j-1}, a_0 > 0, j = 1, 2, \cdots, n$ is a polynomial of degree n, then according to a classical result of Eneström-Kakeya, all the zeros of $P(z)$ lie in $|z| \leq 1$. In this paper, we prove some extensions and generalizations of this result.*

Key Words and Phrases: Polynomial, Zeros, Eneström-Kakeya *Theorem*

2000 Mathematics Subject Classifications: 30C10, 30C15

1. Introduction and Statements of Results

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, then concerning the distribution of zeros of $P(z)$, Eneström and Kakeya [10, 11] proved the following interesting result.

Theorem A. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0, \tag{1}$$

then $P(z)$ has all its zeros in $|z| \leq 1$.

In the literature [1-11], there exist several extensions and generalizations of this Theorem. Joyal *et al* [9] extended Theorem A to the polynomials whose coefficients are monotonic but not necessarily non-negative. In fact they proved the following result.

Theorem B. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

then $P(z)$ has all its zeros in the disk

$$|z| \leq \frac{1}{|a_n|} \left(|a_n| - a_0 + |a_0| \right).$$

Whereas Govil and Rahman [8] extended the result to the class of polynomial with complex coefficients by proving the following interesting result.

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Theorem C. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,*

$$|arg\ \ a_j-\beta|\le\alpha\le\frac{\pi}{2},\quad 0\le j\le n$$

and

$$|a_n|\ge|a_{n-1}|\ge\cdots\ge|a_1|\ge|a_0|,$$

then $P(z)$ has all its zeros in the disk

$$|z|\le(Sin\alpha+Cos\alpha)+\frac{2Sin\alpha}{|a_n|}\sum_{j=0}^{n-1}|a_j|.$$

Aziz and Zargar [2] relaxed the hypothesis of Theorem A and proved the following extension of Theorem A.

Theorem D. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k\ge1$,*

$$ka_n\ge a_{n-1}\ge\cdots\ge a_1\ge a_0>0. \tag{2}$$

then $P(z)$ has all its zeros in $|z+k-1|\le k$.

In this paper, we prove some generalizations and extensions of Theorem C and Theorem D and hence of the Eneström-Kakeya Theorem. In this direction we first present the following interesting result which is a generalization of Theorem C.

Theorem 1. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,*

$$|arg\ \ a_j-\beta|\le\alpha\le\frac{\pi}{2},\quad 0\le j\le n,$$

and for $k\ge1, 0\le\rho\le1$,

$$k|a_n|\ge|a_{n-1}|\ge\cdots\ge|a_1|\ge\rho|a_0|,$$

then all the zeros of $P(z)$ lie in

$$\begin{aligned}|z+k-1| \quad &\le \quad \frac{1}{|a_n|}\Bigg\{(k|a_n|-\rho|a_0|)\,(Sin\alpha+Cos\alpha) \\ &+ \quad (2-\rho+2\rho Sin\alpha)\,|a_0|+2Sin\alpha\sum_{j=1}^{n-1}|a_j|\Bigg\}.\end{aligned} \tag{3}$$

Remark 1. For $\rho=1, k=1$, Theorem 1 reduces to Theorem C. Taking $\rho=1$, in Theorem 1, we get the following result.

Corollary 1. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,*

$$|arg\ \ a_j-\beta|\le\alpha\le\frac{\pi}{2},\quad j=0,1,2,\cdots,n$$

and for $k \geq 1$,

$$k|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $P(z)$ *lie in*

$$|z+k-1| \leq \frac{1}{|a_n|}\left\{(k|a_n|-|a_0|)(Sin\alpha + Cos\alpha) + |a_0| + 2Sin\alpha\sum_{j=0}^{n-1}|a_j|\right\}. \qquad (4)$$

Also by taking $k=1$, in Theorem 1, we get the following generalization of Theorem C.

Corollary 2. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients such that for some real* β,

$$|arg\ \ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad 0 \leq j \leq n,$$

and for $0 \leq \rho \leq 1$,

$$|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq \rho|a_0|,$$

then all the zeros of $P(z)$ *lie in*

$$|z| \leq \frac{1}{|a_n|}\left\{(|a_n|-\rho|a_0|)(Sin\alpha + Cos\alpha) + (2-\rho+2\rho Sin\alpha)|a_0| + 2Sin\alpha\sum_{j=1}^{n-1}|a_j|\right\}. \qquad (5)$$

If we take $k=\frac{|a_{n-1}|}{|a_n|} \geq 1$ in Corollary 1, we obtain the following result.

Corollary 3. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients such that for some real* β,

$$|arg\ \ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j=0,1,\cdots,n$$

and

$$|a_n| \leq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $P(z)$ *lie in*

$$\left|z+\frac{|a_{n-1}|}{|a_n|}-1\right| \leq \frac{1}{|a_n|}\left\{(|a_{n-1}|-|a_0|)(Sin\alpha + Cos\alpha) + |a_0| + 2Sin\alpha\sum_{j=0}^{n-1}|a_j|\right\}.$$

Next we present the following result which is also a generalization of Theorem C.

Theorem 2. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients such that for some real* β,

$$|arg\ \ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad 0 \leq j \leq n-1,$$

$$|arg\,(\lambda+a_n)-\beta|\leq\alpha\leq\frac{\pi}{2}$$

and

$$|\lambda+a_n|\geq|a_{n-1}|\geq\cdots\geq|a_1|\geq|a_0|,$$

then for every real or complex number λ, all the zeros of $P(z)$ lie in the disk

$$|z+\frac{\lambda}{a_n}|\leq\frac{1}{|a_n|}\left\{(|\lambda+a_n|-|a_0|)\,(Sin\alpha+Cos\alpha)+|a_0|+2Sin\alpha\sum_{j=0}^{n-1}|a_j|\right\}. \qquad (6)$$

Remark 2. For $\lambda=0$, Theorem 1 reduces to Theorem C and for $\lambda=(k-1)|a_n|, k\geq 1$, it reduces to Corollary 1.

Applying Theorem 2 to the polynomial $P(tz)$, we obtain the following result.

Corollary 4. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients such that for some real β,*

$$|arg\;\,a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\quad 0\leq j\leq n-1,$$

$$|arg\,(\lambda+a_n)-\beta|\leq\alpha\leq\frac{\pi}{2}$$

and for some $\lambda>0, t>0$

$$\lambda+t^n|a_n|\geq t^{n-1}|a_{n-1}|\geq\cdots\geq t|a_1|\geq|a_0|,$$

then $P(z)$ has all its zeros in the disk

$$|z+\frac{\lambda}{t^{n-1}a_n}|\leq\frac{t}{|a_n|}\left\{\left|\frac{\lambda}{t^n}+a_n-\frac{a_{n-1}}{t}\right| + \left(\frac{|a_{n-1}|}{t}-\frac{|a_0|}{t^n}\right)(Sin\alpha+Cos\alpha)\right.$$
$$\left. + \frac{|a_0|}{t^n}+2Sin\alpha\sum_{j=0}^{n-2}|a_j|t^{j-n}\right\}.$$

Instead of proving Theorem 2, we prove the following more generalization.

Theorem 3. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients such that for some real β,*

$$|arg\;\,a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\quad 1\leq j\leq n-1,$$

$$|arg\,(\lambda+a_n)-\beta|\leq\alpha\leq\frac{\pi}{2},\qquad |arg\,(a_0-\mu)-\beta|\leq\alpha\leq\frac{\pi}{2}$$

and

$$|\lambda+a_n|\geq|a_{n-1}|\geq\cdots\geq|a_1|\geq|a_0-\mu|,$$

then for every real or complex numbers λ and μ, *all the zeros of $P(z)$ lie in the disk*

$$|z+\frac{\lambda}{a_n}|\quad\leq\quad\frac{1}{|a_n|}\left\{\,(|\lambda+a_n|-|a_0-\mu|)\,(Sin\alpha+Cos\alpha)+|a_0|\right. \qquad (7)$$

$$+ \quad |\mu| + 2Sin\alpha \sum_{j=1}^{n-1} |a_j| + 2|a_0 - \mu| Sin\alpha \Bigg\}.$$

Remark 3. For $\mu = 0$, Theorem 3 reduces to Theorem 2.

2. Lemma

For the proofs of these theorems, we need the following result due to Govil and Rahman[8].

Lemma. *If* $|arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, *and for some* $t > 0, |ta_j| \geq |a_{j-1}|$, *then*

$$|ta_j - a_{j-1}| \leq \{(|ta_j| - |a_{j-1}|) Cos\alpha + (|ta_j| + |a_{j-1}|) Sin\alpha\}.$$

3. Proofs of the Theorems

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)\left(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right) \\
&= a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 - a_n z^{n+1} - a_{n-1}z^n - \cdots - a_0 z \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0)z + a_0 \\
&= -a_n z^n (z + k - 1) + (ka_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} \\
&+ \cdots + (a_1 - \rho a_0)z + (\rho - 1)a_0 z + a_0.
\end{aligned}$$

This gives

$$\begin{aligned}
|F(z)| &\geq |z|^n \Bigg\{ (|a_n||z+k-1|) - \Bigg(|ka_n - a_{n-1}| \, |z|^n \\
&+ \cdots + |a_1 - \rho a_0| \, |z| + |\rho - 1||a_0||z| + |a_0| \Bigg) \Bigg\} \\
&= |z|^n \Bigg\{ (|a_n||z+k-1|) - \Bigg(|ka_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} \\
&+ \cdots + \frac{|a_1 - \rho a_0|}{|z|^{n-1}} + \frac{(1-\rho)|a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Bigg) \Bigg\}.
\end{aligned}$$

Now, let $|z| \geq 1$, so that $\frac{1}{|z|^{n-j}} \leq 1, 0 \leq j \leq n$, then we have

$$|F(z)| > |z|^n \Bigg\{ |a_n||z+k-1| \quad - \quad \Bigg(|ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}|$$

$$+ \quad \cdots + |a_1 - \rho a_0| + (1-\rho)|a_0| + |a_0|\Big)\Big\}.$$

Using Lemma, we get

$$\begin{aligned} |F(z)| &> |z|^n \Big\{ |a_n||z+k-1| - \Big((k\,|a_n| - \rho\,|a_0|)\,(Sin\alpha + Cos\alpha) \\ &+ (2-\rho+2\rho Sin\alpha)|a_0| + 2Sin\alpha \sum_{j=1}^{n-1} |a_j| \Big)\Big\}. \end{aligned}$$

> 0, if

$$|z+k-1| > \frac{1}{|a_n|}\left\{ (k\,|a_n| - \rho\,|a_0|)\,(Sin\alpha + Cos\alpha) + (2-\rho+2\rho Sin\alpha)|a_0| + 2Sin\alpha \sum_{j=1}^{n-1} |a_j| \right\}$$

Thus all the zeros of $F(z)$ whose modulus is greater than or equal to 1 lie in

$$|z+k-1| \le \frac{1}{|a_n|}\left\{ (k\,|a_n| - \rho\,|a_0|)\,(Sin\alpha + Cos\alpha) + (2-\rho+2\rho Sin\alpha)|a_0| + 2Sin\alpha \sum_{j=1}^{n-1} |a_j| \right\}$$

But those zeros of $F(z)$ whose modulus is less than 1 already satisfy the above inequality. Hence it follows that all the zeros of $F(z)$ lie in

$$|z+k-1| \le \frac{1}{|a_n|}\left\{ (k\,|a_n| - \rho\,|a_0|)\,(Sin\alpha + Cos\alpha) + (2-\rho+2\rho Sin\alpha)|a_0| + 2Sin\alpha \sum_{j=1}^{n-1} |a_j| \right\}$$

Since all the zeros of $P(z)$ are also the zeros of $F(z)$, we conclude that all the zeros of $P(z)$ lie in the disk

$$|z+k-1| \le \frac{1}{|a_n|}\left\{ (k\,|a_n| - \rho\,|a_0|)\,(Sin\alpha + Cos\alpha) + (2-\rho+2\rho Sin\alpha)|a_0| + 2Sin\alpha \sum_{j=1}^{n-1} |a_j| \right\}$$

This completes the proof of Theorem 1.

Proof of Theorem 3. Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= (1-z)\left(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\right) \\ &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 - a_n z^{n+1} - a_{n-1} z^n - \cdots - a_0 z \\ &= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \cdots + (a_1 - a_0) z + a_0 \\ &= -(a_n z + \lambda) z^n + (a_n + \lambda - a_{n-1}) z^n \\ &+ (a_{n-1} - a_{n-2}) z^{n-1} + \cdots + (a_1 - a_0 + \mu) z - \mu z + a_0. \end{aligned}$$

This gives

$$\begin{aligned}|F(z)| &\geq |z|^n\Bigg\{|a_nz+\lambda|-\Bigg(|a_n+\lambda-a_{n-1}|\,|z|^n+|a_{n-1}-a_{n-2}|\,|z|^{n-1}\\ &+\cdots+|a_1-a_0+\mu||z|+\mu|z|+|a_0|\Bigg)\Bigg\}\\ &= |z|^n\Bigg\{|a_nz+\lambda|-\Bigg(|a_n+\lambda-a_{n-1}|+\frac{|a_{n-1}-a_{n-2}|}{|z|}\\ &+\cdots+\frac{|a_1-a_0+\mu|}{|z|^{n-1}}+\frac{\mu}{|z|^{n-1}}+\frac{|a_0|}{|z|^n}\Bigg)\Bigg\}.\end{aligned}$$

Now, let $|z|\geq 1$, so that $\frac{1}{|z|^{n-j}}\leq 1, 0\leq j\leq n$, then we have

$$\begin{aligned}|F(z)|>|z|^n\Bigg[|a_nz+\lambda|-\Bigg\{|a_n+\lambda-a_{n-1}| &+ |a_{n-1}-a_{n-2}|\\ &+\cdots+|a_1-(a_0-\mu)|+\mu+|a_0|\Bigg\}\Bigg].\end{aligned}$$

Using Lemma, we get

$$\begin{aligned}|F(z)|\geq|z|^n\Bigg\{|a_nz+\lambda| &- \Bigg((|\lambda+a_n|-|a_0-\mu|)(Cos\alpha+Sin\alpha)+|a_0|\\ &+ |\mu|+2|a_0-\mu|Sin\alpha+2Sin\alpha\sum_{j=1}^{n-1}|a_j|\Bigg)\Bigg\}.\end{aligned}$$

>0, if

$$\begin{aligned}|a_nz+\lambda| &> \Bigg\{(|\lambda+a_n|-|a_0-\mu|)(Cos\alpha+Sin\alpha)+|a_0|\\ &+ |\mu|+2|a_0-\mu|Sin\alpha+2Sin\alpha\sum_{j=1}^{n-1}|a_j|\Bigg\}.\end{aligned}$$

i.e, if

$$\begin{aligned}\left|z+\frac{\lambda}{a_n}\right| &> \frac{1}{|a_n|}\Bigg\{(|\lambda+a_n|-|a_0-\mu|)(Cos\alpha+Sin\alpha)+|a_0|\\ &+ |\mu|+2|a_0-\mu|Sin\alpha+2Sin\alpha\sum_{j=1}^{n-1}|a_j|\Bigg\}.\end{aligned}$$

Thus all the zeros of $F(z)$ whose modulus is greater than or equal to 1 lie in

$$\left|z+\frac{\lambda}{a_n}\right| \leq \frac{1}{|a_n|}\Bigg\{(|\lambda+a_n|-|a_0-\mu|)(Cos\alpha+Sin\alpha)+|a_0|$$

$$+ \quad |\mu| + 2|a_0 - \mu| Sin\alpha + 2Sin\alpha \sum_{j=1}^{n-1} |a_j| \Bigg\}.$$

But those zeros of $F(z)$ whose modulus is less than 1 already satisfy the above inequality. Hence it follows that all the zeros of $F(z)$ lie in

$$\left| z + \frac{\lambda}{a_n} \right| \leq \frac{1}{|a_n|} \Bigg\{ (|\lambda + a_n| - |a_0 - \mu|)(Cos\alpha + Sin\alpha) + |a_0| + |\mu| + 2|a_0 - \mu| Sin\alpha + 2Sin\alpha \sum_{j=1}^{n-1} |a_j| \Bigg\}.$$

Since all the zeros of $P(z)$ are also the zeros of $F(z)$, we conclude that all the zeros of $P(z)$ lie in the disk

$$\left| z + \frac{\lambda}{a_n} \right| \leq \frac{1}{|a_n|} \Bigg\{ (|\lambda + a_n| - |a_0 - \mu|)(Cos\alpha + Sin\alpha) + |a_0| + |\mu| + 2|a_0 - \mu| Sin\alpha + 2Sin\alpha \sum_{j=1}^{n-1} |a_j| \Bigg\}.$$

This completes the proof of Theorem 3.

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