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On estimates for the number of sheets of coverings defined by the system of equations

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Abstract. We consider the general case of manifolds of an arbitrary nature given by a system of equations with definite smoothness, and investigate the local behaviour of the manifold. The research demonstrates that the number of covering sheets is determined by the interaction between the local behavior of the manifold—characterized by the minors of the Jacobian matrix—and the linear dimensions of the domain containing the manifold. By applying the implicit function theorem and analysing the structure of tensor fields, the paper provides a series of upper bounds:

Key Words and Phrases: Manifolds, algebraic variety, Jacobian matrix, singular numbers, number of sheets

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1. Introduction

In the literature [1], estimates for the number of covering sheets is studied in connection with fundamental groups acting on the manifold. There, manifolds are studied by the methods of algebraic topology. We shall not use algebraic methods. There is a great gap between algebraic varieties and manifolds defined by maps to $\mathbf{R}^n$, in principle. The main difference is caused by that the algebraic varieties are defined in Zariski topology. This is a topology different from topologies that are commonly used in real or complex analysis, and it is not Hausdorff. The system of algebraic equations is included in the definition of the notion of algebraic manifold. To show the equivalence of these two notions, it is necessary to solve the system of equations in open sets, which demands the satisfaction of definite conditions. Moreover, we shall consider the more general case of manifolds which may be non-algebraic.

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Estimates for the number of sheets are very valuable in applications where the manifolds defined by the system of equations arise. The basic tool in studying of manifolds, given by the system of equations, serves the theorem on implicit functions. But this theorem defines the manifold locally. By this reason, the number of sheets demands special methods of investigation, due to its global character. In the works mentioned above [2], two theorems are proven on the number of sheets of coverings.

Note that in algebraic geometry [3], the number of connected components of algebraic varieties was also investigated. The number of components has a weak connection to the number of covering sheets and is not so substantive in metric questions, investigated here. Moreover, developed methods concern the algebraic varieties only.

There are principal differences between definitions of algebraic varieties and manifolds defined by maps into the space $\mathbf{R}^n$ (we shall call it the functional definition). To establish, in the concrete situation, the equivalence of two notions, it required the solution of the system of equations. But the question is sensitive, and demands definite conditions. Moreover, our objects of investigation are general manifolds which may not be algebraic. If the maps defining coverings are not bijective, the question of the number of sheets naturally arises.

In this work, we consider arbitrary manifolds of definite smoothness. Note that one of the essential parameters, influencing the number of sheets, are the linear sizes of domains, in which the system of equations defining the manifold is given. Local behavior of manifolds is determined by minors of the Jacobi matrix of the system of functions, entering to the left hand sides of equations defining the manifold. The number of sheets are defined by two factors: local behavior of the minors and the sizes of domains (see the example below).

Main tools in our investigations are the results established in [4,6,9,10,11], devoted to the study of the structures of tensor fields. In these works, the metric questions of the theory of manifolds, defined by the system of equations, are studied. The established results demand new angle of a view to the theory of surface integrals, extended to the manifolds serving the solution for the system of equations. There, the surface integrals are defined not in ordinary meaning, but in some improper sense, which eliminates difficulties connected with very complex structure of intersection of the manifolds with the boundary of the Jordan domain, containing it. The obtained results show an actuality some questions of the Riemann integral theory, despite that this theory is completed over 100 years ago.

2. Basic notions and auxiliary results

To pass to considering the main results, we briefly mention some basic notions.

Let us consider two regular manifolds of equal dimensions M and N. $f : M \to N$ is some their map.

Definition 1. This map is called a covering if the following conditions are satisfied:

1. The Jacobian of the map f is distinct from zero in every point of the manifold M.

2. For every point $y \in N$ there exists a neighborhood $y \in U \subset N$ such that the preimage

$$f^{-1}(U) \subset M$$

consists of finite or infinite number of non-intersecting domains

$$f^{-1}(U) = V_1 \bigcup V_2 \bigcup \cdots ,$$

for which the map $f : V_j \to U$ is diffeomorphism;

3. The set M is covered by a finite or denumerable family of such domains U.

The manifold N is called a base of the covering, and M is called to be space of the covering.

In the cases of algebraic or analytic manifolds, the question of estimating the number of sheets is investigated, due to coverings' connections with some groups of transformations of manifolds [1]. In the works mentioned above, there were proven following two theorems were proven, concerning estimates for the number of sheets of coverings. For the formulation of these theorems, we need some notations.

Suppose we are given some closed Jordan domain Ω, included in other open domain Ω_0 in n-dimensional space $\mathbf{R}^n$. Suppose that in the domain Ω_0, some continuous function $f(\bar{x}) = f(x_1, ..., x_n)$ and continuously differentiable functions

$$f_j(\bar{x}) = f_j(x_1, ..., x_n),\ j = 1, ..., r,\ r < n$$

are given. Let the Jacobi matrix

$$\frac{\partial(f_1, ..., f_r)}{\partial(x_1, ..., x_n)}$$

to have everywhere in Ω_0 maximal rank. Consider in Ω the system of equations

$$f_j(x_1, ..., x_n) = 0,\ j = 1, ..., r,\ r < n. \qquad (1)$$

Then in some neighbourhood of an arbitrary inner point of the domain Ω, this system defines a manifold of dimension $n-r$. Later we call the Jacobi matrix of this system of functions as the Jacobi matrix of the system of equations (1).

How we can define the covering considering the system of equations? Let us examine, for this purpose, an example. Take in the open rectangle $(0,4\pi)\times(1/4,3/4)$ the equation $y^2+\cos^2 x=1$. This equation defines manifolds given by the equalities $y=\pm\sin x$ (we suffice with the sign '+'), being taken in different intervals : $(\arcsin(1/4),\ \arcsin(3/4))$,$(\pi-\arcsin(3/4),\ \pi-\arcsin(1/4))$,$(2\pi+\arcsin(1/4),\ 2\pi+\arcsin(3/4))$, $(3\pi-\arcsin(3/4),\ 3\pi-\arcsin(1/4))$. The same map $y=\sin x$, in agree with the definition above, defines a covering with the base $(1/4,3/4)$. Passing, from one part of the manifold to other one, can be performed by continuously moving the variable x. This situation arises due to the periodicity of the function $y=\sin x$. In general, a similar situation can occur in non-periodic cases.

Consider the general case of the systems of the form (1). Denote by Π the set of solutions of this system. Taking some minor of the Jacobi matrix, suppose that the columns of this matrix are placed on the first r columns, and it is distinct from zero in some subdomain of Ω. Suppose that the projection of this subdomain into the subspace $\boldsymbol{R}^{n-r}$, includes some cube B. This projection defines the covering $\pi:\Pi\to B$, by applying the theorem on implicit functions and solving the system. In algebraic geometry, where the basic field is supposed to be algebraically closed, the number of sheets of this covering is studied by a group of transformations, defined on Π. In the general case, when the main field is real, we shall use other methods of investigation.

Theorem 1. Suppose the conditions above are satisfied and the system of equations defines some covering with the base B, $B\subset\boldsymbol{R}^{n-r}$ being a cube. Then, for the number of sheets $|\Gamma|$ of covering, the following inequality is satisfied

$$|\Gamma|\le|B|^{-1}\int_{\Pi}ds,$$

where Π denotes the surface of solutions for the system (1), and $|B|$ denotes the volume of the domain B.

This theorem is a result of the type of work [2], and it gives an answer to the given question. But in the right-hand side, the surface integral, extended to the manifold of solutions, stands. The following theorem, in some cases, may be more useful.

Theorem 2. In the conditions of Theorem 1, the following inequality holds true for the number of covering sheets:

$$|\Gamma| \leq |B|^{-1} \lim_{h\to 0} \frac{1}{h^r} \int_{\substack{0 < f_1 < h \\ \cdots \\ 0 < f_r < h}} \sqrt{G} dx_1 \cdots dx_n.$$

Proof of Theorems 1 and 2. Consider the surface integral

$$\int_{\Pi} ds,$$

extended to the surface Π, defined by the system of equations (1). Surface element on Π has a view $(\sqrt{G}/M) du_1 \cdots du_{n-r}$, and here M denotes the minor, distinct from zero, allowing solve the system (1) with respect to some r variables, denoting free variables in the base B by $u_1, ..., u_{n-r}$. Trivially, we have the inequality $\sqrt{G}/M \geq 1$. Since the system of equations (1) defines a manifold with the base B, we can separate exactly $|\Gamma|$ number of parts (sheets) of the surface Π, and write

$$|\Gamma|\,|B| \leq \sum_{j \leq |\Gamma|} \int_{B_j} \frac{\sqrt{G}}{M} dx_{r+1} \cdots dx_n \leq \int_{\Pi} ds = \int_{\Pi} \sqrt{G} \frac{ds}{\sqrt{G}},$$

where B_j for $j = 1, \ldots, |\Gamma|$, denotes the parts of the manifold Π, being different preimages of the covering with the base B. Therefore, Theorem 1 is true. Applying Lemma 3 [4,6,9,10,11], we can rewrite the relation above as follows:

$$|\Gamma| \leq |B|^{-1} \lim_{h\to 0} \frac{1}{h^r} \int_{\substack{0 < f_1 < h \\ \cdots \\ 0 < f_r < h}} \sqrt{G} dx_1 \cdots dx_n.$$

The relation of Theorem 2 is proven.

As it is seen from the formulation of Theorem 2, in the right-hand side of the inequality, the integral taken along a n-dimensional part of the space stands, instead of the surface integral of Theorem 1. Estimates of such an integral can be reduced to the estimates of volumes in n-dimensional space, where the estimates of the work [4] on the structure of tensor fields can be useful.

3. Estimates involving the linear sizes

From the said above it stands clear that we study manifolds in some wider domain (the dimension of which is greater than the dimension of the manifold), including all solutions of the system. By this reason, naturally, the number of covering sheets defined by this system of equations depends on size of the considered domain. To clarify this statement, consider, for example, the manifold given by a simple equation. Take in the rectangle $(0, a) \times (0, 1/2)$ the manifold defined by the equation $y^2 + \cos^2 x = 1$. Solving this equation with respect to y, we find one of its solutions: $y = \sin x$. This map defines a covering with the base (0, 0.5) (for this purpose, we can take any interval from the segment [-1, 1]). The number of sheets depends on the parameter a, that is, on the size of the domain in which we consider the given equation. If, instead of this equation, considering the equation $y - \sin 2x = 0$, then the number of corresponding covering sheets will be dependent on the multiplier 2 (the number of oscillations will increase), apart from the sizes of the domain in which we consider the minors of the Jacobi matrix, that is, the components of the vector $(-2\cos 2,\, 1)$.

These notes give a scheme for estimating the number of sheets. From theorems 1 and 2, it is clear that the upper bound for this number (during the base of the covering remains unchanged) depends on the value of the maximal minor of the Jacobi matrix and the linear sizes of the domain. We have to estimate the volume of the domain in which the values of the functions, standing in the left-hand sides of the system, fall into determined segments.

To formulate our basic results on the number of sheets, consider the system of equations (1) and suppose that it defines a covering with the base B. We suppose that the Jacobi matrix of the system (1) has in Ω maximal rank. Note that surface integral of the theorem 1 is extended to all solutions of the system (1). By this reason, the sheets of the covering defined by this system take part in the integration along the surface.

Theorem 3. Let the conditions and designations of Theorem 1 be satisfied. Suppose that the domain Ω is enclosed in the cube $[0, K]^n$. Then, for the number of sheets defined by the system of equations (1) the inequality

$$|\Gamma| \leq \begin{pmatrix} n \\ r \end{pmatrix}^{3/2} |B|^{-1} K^{n-r}$$

holds true.

Proof. We begin to prove the theorem with the notes below. Solving the system (1) we parameterize every sheet by various values of independent variables remaining after of application of the theorem on implicit functions. For example, if in some subdomain the minor placed in the first columns of the Jacobi matrix, is distinct from zero, then in this subdomain the system is possible to solve with respect to the variables $x_1, ..., x_r$, moreover, due to uniqueness of the solution we get one sheet parameterized by other independent variables $x_{r+1}, ..., x_n$ in some subdomain of the space $\mathbf{R}^{n-r}$. We can accept that the variables $x_{r+1},...,x_n$ are varying in the base of the covering. By analogy, we define another part of the solution, that is, another sheet taking a new subdomain of variation of the variables $x_{r+1},...,x_n$. By compactness, we find a finite number of subdomains in $\mathbf{R}^{n-r}$ parameterising all sheets of the covering.

In the conditions of the theorem, in every point of the domain Ω, one of the minors of maximal rank is distinct from zero. Therefore, the domain Ω can be dissected into no more than $l = \begin{pmatrix} n \\ r \end{pmatrix}$ connected subdomains, in every of which one of the minors of the Jacobi matrix takes maximal absolute values among all other minors. Denote these subdomains as $\Omega_1,\ldots,\Omega_l$. Then, the surface integral of Theorem 1 can be represented as a sum of surface integrals extended to these subdomains:

$$\int_B ds = \int_{B,\Omega_1} ds + \cdots + \int_{B,\Omega_l} ds.$$

Supposing that the first integral in the right-hand side has a maximal value. So, we have

$$\int_B ds \leq l \int_{B,\Omega_1} ds.$$

The surface element ds in the subdomain Ω_i can be represented as $(\sqrt{G}/M_i)du_{i1}\cdots du_{in-r}$, where M_i denotes the maximal minor in the subdomain Ω_i and $u_{i1},....,u_{in-r}$ are denoting independent variables in this subdomain. Since Gram's determinant can be represented as a sum of squares of all minors ([5]), then we have

$$G = \sum_j G_j^2,$$

where G_j, $j = 1,...,l$ denotes the minors of the Jacobi matrix. So, denoting maximal minor as M, we obtain

$$G = \sum_j G_j^2 \leq lM^2.$$

Consequently,

$$\sqrt{G}M^{-1} \leq \sqrt{\begin{pmatrix} n \\ r \end{pmatrix}}.$$

Then, denoting the minimal value of Gram's determinant by G_0, we have

$$\int_{B,\Omega_1} ds = \int_{B,\Omega_1} \sqrt{G}M^{-1} du_{i1}\cdots du_{in-r}$$

$$\leq \sqrt{\binom{n}{r}} \int_0^K du_{i1} \cdots du_{in-r} \leq K^{n-r} \sqrt{\binom{n}{r}}.$$

Therefore,

$$\int_B ds \leq lK^{n-r} \sqrt{\binom{n}{r}}.$$

So, we have

$$|\Gamma| \leq \binom{n}{r}^{3/2} |B|^{-1} K^{n-r}.$$

Theorem 3 is proven.

4. Estimates involving singular numbers

Denote by $f_{ij} = \partial f_i / \partial x_j$. The matrix $A_0 = (f_{ij})$ is a Jacobi matrix of the system (1). Arranging the entries of all columns consequently in a line, we take the Jacobi matrix of the resulting system of functions, denoting it as A_1 (see [4]). Define the parameter L as a maximal value for the Euclidean norms of the matrices A_0 and A_1. Denote by G_1, as in [4], the minimal value of the product of last $n - r$ singular numbers of the matrix A_1, when they are arranged in descending order. Now we prove new bound for the number of sheets, supposing that all the functions considered above are continuously differentiable in Ω_0.

Theorem 4. In conditions above, the following bound holds true:

$$|\Gamma| \leq c_r \, |B|^{-1} L^r G_1^{-1}$$

where $c_r = \sum_{s=1}^{\infty} s^{r-1} 2^{-s}$.

Proof. As it was made above, we can dissect the domain Ω into no more than l connected subdomains, in every of which one of minors of the Jacobi matrix takes maximal absolute values among all other minors. Taking one of these subdomains, we suppose that the minor placed at the first r columns of the Jacobi matrix is maximal. Let us make the change of variables under the integral in the relation of Theorem 2, using formulae

$u_1 = f_1(x_1, ..., x_n), \ \ldots, \ u_r = f_r(x_1, ..., x_n), \ u_{r+1} = x_{r+1}, \ \ldots, \ u_n = x_n.$

Jacobian of this exchange is as follows

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_1} & 0 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{\partial f_1}{\partial x_r} & \cdots & \frac{\partial f_r}{\partial x_r} & 0 & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & 1 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & \cdots & 0 & 0 & \cdots & 1 & \end{vmatrix} = M_1.$$

So, we have

$$\int_{\substack{0<f_1<h \\ \cdots \\ 0<f_r<h}} \sqrt{G} dx_1 \cdots dx_n =$$

$$= \int_0^h \cdots \int_0^h \int_{\substack{f_1=u_1 \\ \cdots \\ f_r=u_r}} \sqrt{G} M_1^{-1} du_1 \cdots du_1 dx_{r+1} \cdots dx_n.$$

Consider the inner integral. Let this integral to take the maximal value for some $u_1 = u_1', \ldots, u_r = u_r'$ (we denote by Π_1 the corresponding surface). Then, applying the mean value theorem, we can write:

$$\int_{\substack{0<f_1<h \\ \cdots \\ 0<f_r<h}} \sqrt{G} dx_{r+1} \cdots dx_n = h^r \int_{\substack{f_1=u_1' \\ \cdots \\ f_r=u_r'}} \sqrt{G} M_1^{-1} dx_{r+1} \cdots dx_n.$$

Since the Jacobi matrix has the maximal rank, then the maximal minor M_1 is distinct from zero in the closed domain Ω_1. From the representation of the Gram determinant G as a sum of squares of all minors, it follows that

$$\sqrt{G} M_1^{-1} \leq l^{1/2}.$$

So,

$$\int_{\substack{0<f_1<h \\ \cdots \\ 0<f_r<h}} \sqrt{G} dx_{r+1} \cdots dx_n \leq h^r \sqrt{l} \int_{\substack{f_1=u_1' \\ \cdots \\ f_r=u_r'}} dx_{r+1} \cdots dx_n.$$

For the estimation of the last integral we begin with estimating the volume of the domain defined by the inequality:

$$a_1(x_{r+1}, ..., x_n) ... a_r(x_{r+1}, ..., x_n) \leq H,$$

for positive H; here the $a_j(x_{r+1},...,x_n)$ denotes the j-th singular number of the Jacobi matrix, which we denote by A_0 as in [4, p.87], moreover, the first r coordinates are defined as a functions of $x_{r+1},...,x_n$. Since from linear algebra it is best known, that $a_j(x_{r+1},...,x_n) \leq L, j = 1,...,r$ (in designations from [4, 87]). In [4] (see Corollary 1), it was shown that the map $u_i = a_i(x_{r+1},...,x_n), \quad i = 1,...,r$, is one-to-one. Denote by $W(\bar{u})$ the Jacobi matrix of this transformation. Extending the integration to more wider domain of integration with respect to $\bar{u}$, we obtain:

$$\int_{\Pi_1} dx_{r+1}...dx_n \leq \int_{\substack{0 < u_1...u_r \leq H \\ u_1 \leq ... \leq u_r \leq L}} |\det W(\bar{u})|^{-1} du_1...du_r, \tag{2}$$

where the matrix $W(\bar{u})$ defined by the equality:

$$W(\bar{u}) = \left\| \begin{array}{c} \frac{\partial u_1}{\partial x_{r+1}} \cdots \frac{\partial u_1}{\partial x_n} \\ \cdots\cdots\cdots\cdots \\ \frac{\partial u_r}{\partial x_{r+1}} \cdots \frac{\partial u_r}{\partial x_n} \end{array} \right\|, \quad \bar{u} \in \Pi_1,$$

where $u_1,...,u_r$ are the singular number of the matrix A_0, being looked as a matrix with entries depending on $\bar{u}$. Let the system of vectors $\bar{t}_1(\bar{u}),...,\bar{t}_n(\bar{u})$ and $\bar{q}_1(\bar{u}),...,\bar{q}_{n-r}(\bar{u})$ be singular bases for the matrix A_0 (see [6]), moreover, $\bar{t}_1(\bar{u}) \in R^n, \quad j = 1,...,n, \;\; \bar{q}_i(\bar{u}) \in R^{n-r}, \quad i = 1,...,n-r$.

We have the following equations

$$u_j = (A_0\bar{t}_j, \bar{q}_j), \quad A_0\bar{t}_j = u_j\bar{q}_j, {}^tA_0\bar{t}_j = u_j\bar{t}_j, \tag{3}$$

for $j = 1, \ldots, n-r$. Then

$$\frac{du_j}{\partial x_i} = \left(\frac{\partial A_0}{\partial x_i}\bar{t}_j, \bar{q}_j\right) + \left(A_0\frac{\partial \bar{t}_j}{\partial x_i}, \bar{q}_j\right) + \left(A_0\bar{t}_j, \frac{\partial \bar{q}_j}{\partial x_i}\right) =$$

$$= \left(\frac{\partial A_0}{\partial x_i}\bar{t}_j, \bar{q}_j\right) + \left(\frac{\partial \bar{t}_j}{\partial x_i}, {}^tA_0\bar{q}_j\right) + \left(A_0\bar{t}_j, \frac{\partial \bar{q}_j}{\partial x_i}\right).$$

By using of the equalities (3) and the fact, that

$$\left(\frac{\partial \bar{c}}{\partial x_i}, \bar{c}\right) = 0,$$

(see [1, pp.53]), for every normed vector $\bar{c}(\bar{u})$ we have the substantive relation

$$\frac{du_j}{\partial x_i} = \left(\frac{\partial A_0}{\partial x_i}\bar{t}_j, \bar{q}_j\right) \quad j = 1,...,n-r, \; i = 1,...,n-r,$$

which shows that the vectors of the singular bases behave themselves, at time of differentiation of the singular spectrum, as fixed vectors. Moreover, those relations show that

$$W(\bar{u}) = A'_{0\bar{u}} \cdot T$$

where $A'_{0\bar{u}}$ is a transposed Jacobi matrix of the system of functions, getting by location of the lines of A_0 into a line ([7])

$$f_{11}, \ldots, f_{1n-r}, f_{21}, \ldots, f_{2n-r}, \ldots, f_{n1}, \ldots, f_{nn-r}. \tag{4}$$

Here, f_{ij} are considering as functions of $\bar{u}$, and T is a matrix of the size $n(n-r) \times (n-r)$, consisting of the columns $\bar{t}_1 \otimes \bar{q}_1, ..., \bar{t}_{n-r} \otimes \bar{q}_{n-r}$ lying into the tensor product $\boldsymbol{R}^{n-r} \otimes \boldsymbol{R}^n$. Applying reasoning of the work [4], we obtain

$$(\det W(\bar{u}))^2 = \det\left(A'_{0\bar{u}} \cdot {}^t A'_{0\bar{u}}\right). \tag{5}$$

Denote by $\upsilon_1 \geq \upsilon_2 \geq \cdots \geq \upsilon_{n-r}$ the singular numbers of the matrix $A'_{0\bar{u}}$, (we note that the singular numbers of ${}^t A'_{0\bar{u}}$ and $A'_{0\bar{u}}$ are the same), then by the theorem of Courant and Fisher ([8, p.115]) the $1+i$ –th singular number $\upsilon_i (0 \leq i \leq n-r-1)$ can be represented in the form

$$\upsilon_{i+1}^2 = \min_{V_{r-1}} \max_{\nu \in V_{r-1}} R(\bar{w}), \quad 0 \leq i \leq n-r-1, \tag{6}$$

where the V_{r-i} denote any $n-r-i$ dimensional subspace of R^{n-r}, and $\bar{u}$ –takes all the values from this subspace, and $R(\bar{w})$ is a Relay relation (see [8, p. 107]):

$$R(\bar{w}) = \frac{\left(A'_{0\bar{u}} \cdot {}^t A'_{0\bar{u}} \bar{w}, \bar{w}\right)}{(\bar{w}, \bar{w})}.$$

As it is clear, the matrix ${}^t A'_{0\bar{u}}$ is a Jacobi matrix of the system (4) (the differentiation is taken over $\bar{u}$). Therefore, it can be represented as ${}^t(D(\bar{x}) \cdot {}^t A_1(\bar{x}))$, where $D(\bar{x})$ is a matrix of a view

$$\left\| \begin{array}{lll} \varphi_{11} \cdots \varphi_{1n-r} & 1 & 0 \ldots 0 \\ \varphi_{21} \cdots \varphi_{2n-r} & 0 & 1 \ldots 0 \\ \multicolumn{3}{c}{\dotfill} \\ \varphi_{r1} \cdots \varphi_{rn-r} & 0 & 0 \ldots 1 \end{array} \right\|, \tag{7}$$

and the matrix A_1 is a Jacobi matrix of the system of functions (4) in which now the differentiation is taken with respect to $\bar{x}$. Then $A'_{0\bar{u}} = {}^t A_1(\bar{x}) \cdot {}^t D(\bar{x})$ and putting $\bar{\omega} = {}^t D(\bar{x}) \cdot \bar{w}$ we can represent the relation $R(\bar{w})$ by the following equality

$$R(\bar{w}) = \frac{(A_1 \cdot {}^t A_1 \bar{\omega}, \bar{\omega})}{(\bar{\omega}, \bar{\omega})} \cdot \frac{(\bar{\omega}, \bar{\omega})}{(\bar{w}, \bar{w})} = R_1(\bar{\omega}) \cdot \frac{(\bar{\omega}, \bar{\omega})}{(\bar{w}, \bar{w})}. \tag{8}$$

It is obvious, that $\bar{\omega} \in R^n$ and the lines of the matrix $D(\bar{x})$ are orthogonal to the gradients of the functions$f_i(\bar{x})$ from the system (1) (this is best known from the analysis). The relations (6) now can be written as follows

$$\upsilon_{i+1}^2 = \min_{W_{r-1}} \max_{\bar{\omega} \in W_{r-1}} R_1(\bar{\omega}) \cdot \frac{(\bar{\omega}, \bar{\omega})}{(\bar{w}, \bar{w})},$$

where W_{r-i} is an image of V_{r-i} by the mapping $\bar{\omega} = {}^tD(\bar{x}) \cdot \bar{v}$. Therefore, it is a subspace of a dimension *r-i* in $\boldsymbol{R}^n$, being orthogonal to the gradients of the functions $f_j, j = 1, \ldots, n-r$ and $\bar{\omega}$ is any point of this subspace. From the view of the matrix $D(\bar{x})$(see(7)) it is obvious that

$$(\bar{\omega}, \bar{\omega}) = ({}^tD\bar{w}, {}^tD\bar{w}) = (\bar{w}, \bar{w}) + (D_0\bar{w}, \bar{)} \geq (\bar{w}, \bar{w}), \tag{9}$$

where D_0 is non –negatively defined matrix of the size $r \times r$:

$$D_0 = \begin{Vmatrix} \varphi_{11} & \cdots & \varphi_{1n-r} \\ \cdots & \cdots & \cdots \\ \varphi_{r1} & \cdots & \varphi_{rn-r} \end{Vmatrix} \cdot \begin{Vmatrix} \varphi_{11} & \cdots & \varphi_{r1} \\ \cdots & \cdots & \cdots \\ \varphi_{1n-r} & \cdots & \varphi_{rn-r} \end{Vmatrix}.$$

From (8) and (9) we deduce

$$R'(\bar{w}) \geq R_1(\bar{\omega})$$

and therefore,

$$\upsilon_{i+1}^2 = \min_{W_{r-i}} \max_{\bar{\omega} \in W_{r-i}} \frac{(A_1 \cdot {}^tA_1\bar{\omega}, \bar{\omega})}{(\bar{\omega}, \bar{\omega})}. \tag{10}$$

If we omit the condition of orthogonality of W_{r-i} to the gradients of the functions f_j *j=1,. . . , n-r,* then by that the *min* on the right side of (8) can only stand more less. Consequently, we have

$$\upsilon_{i+1}^2 \geq \min_{W_{r-1}} \max_{\bar{\omega} \in W_{r-1}} \frac{(A_1 \cdot {}^tA_1\bar{\omega}, \bar{\omega})}{(\bar{\omega}, \bar{\omega})},$$

where W'_{r-i} is any $r-i$ dimensional subspace of $\boldsymbol{R}^n$. Then the right-hand side of the last inequality by the theorem of Courant and Fisher gives us $n-r+1+i$–the characteristic number of the matrix$_1 \cdot {}^tA_1$. Therefore,

$$\nu_1 ... \nu_r \geq a'_{n-r+1} ... a'_n,$$

where $a'_1 \geq a'_2 \geq \cdots \geq a'_n$ are the singular numbers of the matrix A_1 (in particular the singular numbers of the matrix $A'_{0\bar{u}}$ are non–zero). So we have proved the inequality

$$|\det W(\bar{u})| \geq G_1(\bar{x}).$$

From (7) we get

$$\int_{\Pi_1} d\xi_1 ... d\xi_r \leq G_1^{-1} \int_{\substack{0 < u_1 ... u_r \leq H \\ u_1 \leq ... \leq u_r \leq L}} du_1 ... du_r.$$

The integral on the right side we dissect into sums of integrals of the form

$$I_s = \int_{\substack{H2^{-s} < u_1 ... u_r \leq H2^{1-s} \\ u_1 \leq ... \leq u_r \leq L}} du_1 ... du_r, \quad s = 1, 2 ....$$

We have $H2^{-s} < u_1 L^{r-1}$, or $u_1 \geq H2^{-s} L^{1-r}$. Hence,

$$I_s \leq \int_{H2^{-s}L^{1-r}}^{L} du_1 ... \int_{H2^{-s}L^{1-r}}^{L} du_{r-1} \int_{H2^{-s}u_1^{-1}...u_{r-1}-1}^{H2^{1-s}u_1^{-1}...u_{r-1}^{-1}} du_r =$$

$$= H2^{-s} (\log 2^s L^r H^{-1})^{r-1}.$$

From the said above it follows that we can take $H = L^r$. Then we represent the estimate found above as follows:

$$I_s \leq L^r 2^{-s} s^{r-1}.$$

By summing over the all n, we obtain the estimation:

$$\int_{\Pi_1} d\xi_1 ... d\xi_r \leq c_r L^r G_1^{-1},$$

where $c_r = \sum_{s=1}^{\infty} s^{r-1} 2^{-s}$. It is useful to note that the estimate of Theorem 3 is appropriate when the system of equations defining the manifold contains periodic functions. Here, the linear sizes play an essential role. The conditions of Theorem 4 are more restrictive, and the linear size here is not so substantive.

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