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A Theorem on the Oscillation of Solutions to Nonuniformly Degenerate Second-Order Elliptic–Parabolic Equations

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Abstract. A class of second-order elliptic–parabolic equations of non-divergence structure with nonuniform power degeneration is considered in the paper. A theorem on the oscillation of solutions of these equations, as well as an interior a priori estimate of the Hölder norm, is proved.

Key Words and Phrases: nonuniformly degenerate, oscillation solutions, elliptic-parabolic equation

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Let E_n and R_{n+1} be Euclidean spaces of points $x = (x_1, \ldots, x_n)$, $n \geq 1$ and $(x,t) = (x_1, \ldots, x_n, t)$,respectively, $\Omega \subset E_n$ be a bounded domain with the boundary $\partial\Omega$ of class C^2, $Q_T = \Omega \times (-T, 0)$ be a cylinder of the given height $T > 0$, $\partial Q_T = \partial\Omega \times [-T, 0]$ be a lateral surface of the cylinder Q_T and $\Gamma(Q_T) = \{(x,t) | x \in \Omega, t = -T\} \cup (\partial\Omega \times [-T, 0])$ is a parabolic boundary of Q_T. In Q_T consider an elliptic-parabolic equation

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x,t) u_{ij} + \varphi(0 - t) u_{tt} - u_t = 0. \tag{1}$$

Here, $\|a_{ij}(x,t)\|$ is a real symmetric matrix, moreover for all $(x,t) \in Q_T$

$$\inf_{Q_T} \sum_{i=1}^{n} \frac{a_{ii}(x,t)}{\lambda_i(x,t)} = \gamma, \tag{2}$$

$$\sigma \equiv \sup_{Q_T} \left[\sum_{i,j=1}^{n} \frac{a_{ij}^2(x,t)}{\lambda_i(x,t)\lambda_j(x,t)} \Bigg/ \left(\sum_{i=1}^{n} \frac{a_{ii}(x,t)}{\lambda_i(x,t)} \right)^2 \right] - \frac{1}{n - e^2} < 0, \tag{3}$$

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$$\varphi(z) \in C^1[-T,0],\ \varphi(z)\ge 0,\ \varphi'(z)\ge 0,$$
$$\varphi(0)=0, \varphi'(0)=0,\ \varphi(z)\ge \beta_1 z\cdot\varphi'(z), \beta_1>0 - \text{is a constant.} \quad (4)$$

Here $\gamma \in (0,1]$ is a constant, $u = u(x,t)$, $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $i,j = 1,\dots,n$, $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $e = \inf\limits_{Q_T}\sum\limits_{i=1}^{n}\frac{a_{ii}(x,t)}{\lambda_i(x,t)} \Big/ \sup\limits_{Q_T}\sum\limits_{i=1}^{n}\frac{a_{ii}(x,t)}{\lambda_i(x,t)}$, $\lambda_i(x,t) = \left[\frac{\omega_i^{-1}(\rho(x)+\sqrt{|t|})}{\rho(x)+\sqrt{|t|}}\right]^2$, $i = 1,\dots,n$, $\rho(x) = \sum\limits_{i=1}^{n}\omega_i(|x_i|)$. At that $\omega_i(z)$ are strongly monotonically increasing functions for $z \in [0, \text{diam}Q_T]$, $\omega_i^{-1}(z)$ are the function inverse to $\omega_i(z)$, besides for $i = 1,\dots,n$ and sufficiently small z

$$\alpha_1\cdot\omega_i(Z) \le \omega_i(\eta\cdot z) \le \alpha_2\cdot\omega_i(Z), \quad (5)$$

$$\left(\frac{\omega_i^{-1}(z)}{z}\right)^{q-1}\cdot\int\limits_0^{\omega_i^{-1}(z)}\left(\frac{\omega_i(\tau)}{\tau}\right)^q d\tau \le A\cdot z, \quad (6)$$

where $\alpha_1 > 1$, $\alpha_2 > 1$, $\eta > 0$, $A > 0$ and $q > n$ are some constants.

The aim of the paper is to prove an inner a priori estimate of Hòlders norm of solutions of the equations (1). Note that the analogous result for second order parabolic and elliptic equations of non-divergent structure has been obtained in papers [1, 3, 4, 10, 11, 12, 14, 17]. Note that the analogous result for uniform parabolic equations of divergent structure was obtained in [2, 15, 16]. A more complete review of results on this theme one can find in [17, 5, 6, 7, 8, 9, 13].

First, we agree on some notation and definitions. We shall denote by $E_R^{x^0}(k)$ the ellipsoid $\left\{x : \sum\limits_{i=1}^{n}\frac{(x_i-x_i^0)^2}{(\omega_i^{-1}(R))^2} < k^2\right\}$, $C_{R:k}^{t^1,t^2}(x^0)$ -a cylinder $E_R^{x^0}(k)\times(t^1,t^2)$. Here $R > 0$, $k > 0$, $t^1 < t^2$, $x^0 \in E_n$. The function $u(x,t) \in C^{2,2}(Q_T)$ is called L-subelliptic-parabolic in Q_T, if $Lu(x,t) \ge 0$ for $(x,t) \in Q_T$. A function $u(x,t)$ is called L-super elliptic-parabolic in Q_T, if -$u(x,t)$ L-subelliptic-parabolic in Q_T.

Let $C^1 = C_{R:17}^{-\frac{9bR^2}{8},0}(0)$, $C^2 = C_{R:1}^{-\frac{bR^2}{16},0}(0)$, $C^3 = C^1 \setminus C^2$, where the constant $b \in (0,1)$ will be chosen later.For $S > 0$ and $\beta > 0$ introduce the function

$$G_R^{S,\beta}(x,t) = \begin{cases} t^{-S}\cdot\exp\left[-\frac{1}{4\beta t}\sum\limits_{i=1}^{n}\frac{x_i^2}{(\omega_i^{-1}(R))^2}\right], & t>0,\\ 0 & t\le 0.\end{cases}$$

The measure μ, determined in B-set $E \subset C^3$ is called (s,β,R)-admissible, if $\int\limits_E G_R^{S,\beta}(x-y,t-\tau)d\mu(\mu,\tau) \le 1$ for $(x,t) \notin E$. The number $P_R^{s,\beta}(E) = \sup\mu(E)$, where the least upper bound is taken on all (s,β,R) admissible measures, is called an elliptic-parabolic s,β,R-capacity of the set E. The record $C(\cdots)$ means that a positive constant C depends only on the content of parenthesis.

Lemma 1. *If the conditions (2)-(6) are fulfilled with respect to the coefficients of the operator L, then there exist constants $S(\gamma,n,b)$, $\beta(\gamma,n,b)$ and $R_0(\gamma,n,b)$ such that for $R\le R_0$, $(y,\tau)\in C^3$*

$$L_{(x,t)}G_R^{s,\beta}(x-y,t-\tau)\ge 0,\quad (x,t)\in C^3\setminus\{(y,\tau)\}. \tag{7}$$

Proof. We have for $t>\tau$ allowing for the conditions (2)-(6). For simplicity, we will denote the function $G_R^{s,\beta}(x,t)$ by $G(x,t$. We have

$$\begin{aligned}
J \;=\; & \frac{LG(x-y,t-\tau)}{G(x-y,t-\tau)}\cdot(t-\tau)= \\
& =\frac{1}{4\beta^2(t-\tau)}\cdot\sum_{i,j=1}^{n}a_{ij}(x,t)\cdot\frac{(x_i-y_i)\,(x_j-y_j)}{(\omega_i^{-1}(R)\,)^2\cdot(\omega_j^{-1}(R)\,)^2}- \\
& -\frac{1}{2\beta}\cdot\sum_{i=1}^{n}\frac{a_{ii}(x,t)}{(\omega_i^{-1}(R)\,)^2}+S-\frac{1}{4\beta(t-\tau)}\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{(\omega_i^{-1}(R)\,)^2}- \\
& -\varphi(0-t)\cdot\frac{S(S+1)}{t-\tau}-\frac{(S+1)\cdot\varphi(0-t)}{2\beta(t-\tau)^2}\cdot\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{(\omega_i^{-1}(R)\,)^2}+ \\
& +\frac{\varphi(0-t)}{16\beta^2\cdot(t-\tau)^3}\cdot\left[\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{(\omega_i^{-1}(R)\,)^2}\right]^2\ge\frac{1}{4\beta^2(t-\tau)}\times \\
& \times\left(\sum_{i,j=1}^{n}\frac{a_{ij}^2(x,t)}{\lambda_i(x,t)\cdot\lambda_j(x,t)}\right)^{1/2}\times \\
& \times\left(\sum_{i,j=1}^{n}\lambda_i(x,t)\lambda_j(x,t)\cdot\frac{(x_i-y_i)^2\cdot(x_j-y_j)^2}{(\omega_i^{-1}(R)\,)^4\cdot(\omega_j^{-1}(R)\,)^4}\right)^{1/2}- \\
& -\frac{1}{2\beta}\cdot\left(\sum_{i=1}^{n}\frac{a_{ii}^2(x,t)}{\lambda_i^2(x,t)}\right)^{1/2}\cdot\left(\sum_{i=1}^{n}\lambda_i^2(x,t)\cdot\frac{1}{(\omega_i^{-1}(R)\,)^4}\right)^{1/2}+ \\
& +S-\frac{1}{4\beta(t-\tau)}\cdot\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{(\omega_i^{-1}(R)\,)^2}+ \\
& +\frac{(s+1)\beta_1\cdot(0-t)\cdot\varphi'(0-t)}{2\beta\cdot(t-\tau)^2}\cdot\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{(\omega_i^{-1}(R)\,)^2}\ge
\end{aligned}$$

$$\geq \frac{1}{4\beta^2(t-\tau)} \cdot \sum_{i=1}^{n} \frac{a_{ii}(x,t)}{\lambda_i^{(}x,t)} \cdot \sum_{i=1}^{n} \lambda_i(x,t) \cdot \frac{(x_i-y_i)^2}{(\omega_i^{-1}(R)\,)^4} -$$
$$-\frac{1}{2\beta} \cdot \sum_{i=1}^{n} \frac{a_{ii}(x,t)}{\lambda_i^{(}x,t)} \cdot \sum_{i=1}^{n} \frac{\lambda_i(x,t)}{(\omega_i^{-1}(R)\,)^2} +$$
$$+s - \frac{1}{4\beta(t-\tau)} \cdot \sum_{i=1}^{n} \frac{(x_i-y_i)^2}{(\omega_i^{-1}(R)\,)^2}, \tag{8}$$

on the other hand for $(x,t) \in C^3$, $|x_i| \leq 17 \cdot \omega_i^{-1}(R)$, $i = 1,\ldots,n$, $\omega_i(|x_i|) \leq \omega_i(17\omega_i^{-1}(R)) \leq \alpha_2 \cdot R$, $\rho(x) = \sum\limits_{i=1}^{n} \omega_i(|x_i|) \leq n\alpha_2 \cdot R$, $\sqrt{|t|} \leq \sqrt{\frac{9b}{8}} \cdot R \leq 2R$ therefore

$$\rho(x) + \sqrt{|t|} \leq (n \cdot \alpha_2 + 2)R. \tag{9}$$

Analogously, if $(x,t) \in C^3$, then either $\sum\limits_{i=1}^{n} \frac{x_i^2}{(\omega_i^{-1}(R))^2} \geq 1$ or $|t| \geq \frac{bR^2}{16}$. In the first case there will be found such i_0, $1 \leq i_0 \leq n$ that $|x_{i_0}| \geq \frac{1}{\sqrt{n}}\omega_{i_0}^{-1}(R)$ or $|t| \geq \frac{\sqrt{b}}{4}R$. We conclude that

$$c_1(n,b) \cdot \left(\frac{\omega_i^{-1}(R)}{R}\right)^2 \leq \lambda_i(x,t) \leq c_2(n,b) \cdot \left(\frac{\omega_i^{-1}(R)}{R}\right)^2, \quad i = 1,\ldots,n. \tag{10}$$

Allowing for (10) in (8) we get

$$J \;\geq\; \frac{\gamma}{4\beta^2(t-\tau)} \cdot \frac{c_1}{R^2} \cdot \sum_{i=1}^{n} \frac{(x_i-y_i)^2}{(\omega_i^{-1}(R))^2} -$$
$$-\frac{n}{\gamma \cdot 2\beta} \cdot \frac{c_2}{R^2} + S - \frac{1}{4\beta(t-\tau)} \cdot \sum_{i=1}^{n} \frac{(x_i-y_i)^2}{(\omega_i^{-1}(R))^2} \geq$$
$$\geq \frac{1}{4\beta(t-\tau)} \cdot \left(\frac{\gamma \cdot c_1}{\beta} - 1\right) \cdot \sum_{i=1}^{n} \frac{(x_i-y_i)^2}{(\omega_i^{-1}(R))^2} + S - \frac{nc_2}{2\gamma\beta}. \tag{11}$$

Now it suffices assume

$$\beta = \gamma c_1, \quad s = \frac{nc_2}{2\gamma^2 \cdot c_1} \tag{12}$$

and the required inequality (7) follows from (11) - (12).

◀

In what follows, unless otherwise specified, we assume that the constants s and β are chosen in correspondence with the equalities (12). For brevity, we shall denote the function $G_R^{s,\beta}(x,t)$ and $P_R^{s,\beta}$ by G_R and P_R respectively.

Lemma 2. *Let* $B = C^{t^0-\rho^2R^2,t^0}_{R:\rho}(x^0)$, $\overline{B} \subset C^3$, $\rho > 0$, *and* $R \in (0,1]$. *Then*

$$c_3(s,\beta)\cdot(\rho R)^{2s} \le P_R(B) \le c_4(s,\beta)\cdot(\rho R)^{2s}. \tag{13}$$

Proof. Let $W(x,t) = G(x,t;x^0,t^0-\rho^2R^2)$, $(x,t) \in \overline{B}$. If $t \le t^0 - \rho^2R^2$, then $W(x,t) = 0$. If $x \notin E^{x^0}_R(\rho)$, then

$$W(x,t) = (t-t^0+\rho^2R^2)^{-s} \times \exp\left[-\frac{1}{4\beta(t-t^0+\rho^2R^2)}\cdot\sum_{i=1}^{n}\frac{(x_i-x_i^0)^2}{(\omega_i^{-1}(R))^2}\right] \le$$
$$\le (t-t^0+\rho^2R^2)^{-s}\cdot\exp\left[-\frac{\rho^2}{4\beta(t-t^0+\rho^2R^2)}\right].$$

For $v > 0$ consider the function $Z(v) = v^{-s}\cdot\exp\left[-\frac{\rho^2}{4\beta v}\right]$, and find the value at which $z(v)$ attains its maximum. We find from the equation $Z'(v) = 0$, $v = \frac{\rho^2}{4\beta s}$. If $t = \frac{\rho^2}{4\beta s} + t^0 - \rho^2R^2$, $x \notin E^{x^0}_R(\rho)$, then

$$W(x,t) = \left(\frac{\rho^2}{4\beta s}\right)^{-s}\cdot\exp(-s) = (4\beta s)^s\cdot(\rho R)^{-2s}\cdot R^{2s}\cdot\exp(-s). \tag{14}$$

Let $t \ge t^0$ and $\rho > 0$, $t - t^0 + \rho^2R^2 \ge \rho^2R^2$, then $(t-t^0+\rho^2R^2)^{-s} \le (\rho R)^{-2s}$

$$\exp\left[-\frac{1}{4\beta(t-t^0+\rho^2R^2)}\cdot\sum_{i=1}^{n}\frac{(x_i-x_i^0)^2}{(\omega_i^{-1}(R))^2}\right] \le 1,$$

and

$$W(x,t) \le (\rho R)^{-2s}. \tag{15}$$

We conclude from (14) and (15) that

$$\sup_{R_{n+1}\setminus\overline{B}} W(x,t) \le a\cdot(\rho R)^{-2s}, \tag{16}$$

where $a = \max = \left\{1;(4\beta s)^s\cdot e^{-s}\cdot R^{2s}\right\}$. Consider the measure $\mu(y,\tau)$ concentrated at the center of the lower foundation of a cylinder B with density $\frac{1}{a}\cdot(\rho R)^{2s}$. If $(x,t) \notin B$,

$$\int_B G(x,y;t,\tau)\,d\mu\,(y,\tau) = \int_{\{x^0,t^0-\rho^2R^2\}} W(x,t)\,d\mu\,\left\{x^0,t^0-\rho^2R^2\right\} \le$$
$$\le a\cdot(\rho R)^{-2s}\cdot\mu\left\{(x^0,t^0-\rho^2R^2)\right\} = 1.$$

Consequently $P_R(B) = \sup \mu(B)$ and

$$P_R(B) \geq \mu(B) = \mu\left\{(x^0, t^0 - \rho^2 R^2)\right\} = \frac{1}{a} \cdot (\rho R)^{2s},$$

and the required estimate (13) is proved.

◀

Let $C^4 = C_{R:9}^{-\frac{bR^2}{8},0}(0)$, $(x^0, t^0) \in \Gamma(C^4)$, $C^5 = C^5(x^0, t^0) = C_{R:8}^{t^0 - bR^2, t^0}(x^0)$, $C^6 = C^6(x^0, t^0) = C_{R:1}^{t^0 - \frac{bR^2}{4}, t^0}(x^0)$, $C^7 = C^7(x^0, t^0) = C_{R:1}^{t^0 - bR^2, t^0 - \frac{bR^2}{2}}(x^0)$. Through $S(C)$ will denote the lateral surface of the cylinder C, and through $F(C)$ is its lowest base.

Choose and fix b so that the condition

$$b\beta s \leq \frac{49}{4} \tag{17}$$

be fulfilled.

Lemma 3. *Let a domain D be contained in the cylinder C^5 and have limit points on $\Gamma(C^5)$ and intersecting C^6. Then let continuous in $\overline{D}$, and vanishing in $\Gamma = \Gamma(D) \cap C^5$ positive L-subelliptic-parabolic function $u(x,t)$ be determined in D. Then if as to the coefficients of the operator L the conditions (2)-(6) are fulfilled, there exist such $\eta_1(\gamma, n)$ that for $R \leq R_0$)*

$$\sup_D \geq \left(1 + \eta_1 \cdot R^{-2s} \cdot P_R(E_R)\right) \sup_{D \cap C^6} u, \quad \textit{where} \tag{18}$$
$$E_R = C^7 \setminus D.$$

Proof. We can consider that $P_R(E_R) > 0$. Fix an arbitrary $\varepsilon \in (0, P_R(E_R))$, and let the measure on E_R, be such that

$$U(x,t) = \int\limits_{E_R} G(x-y, t-\tau) d\mu(y,\tau) \leq 1, \quad (x,t) \notin E_R, \tag{19}$$

$$\mu(E_R) > P_R(E_R) - \varepsilon. \tag{20}$$

Let (y, τ) be an arbitrary fixed point from $E_R, S(C^5)$ a lateral surface of C^5. Now estimate the quantity $\sup\limits_{(x,t) \in S(C^5)} G(x-y, t-\tau)$. To this end we fix $x \in \partial E_R^{x^0}(8)$ and find that value of $t > \tau$, at which the function $v(t) = G(x-y, t-\tau)$ attains its maximum. Setting $v'(t)$ to zero, we get

$$t - \tau = \frac{1}{4\beta s} \cdot \sum_{i=1}^{n} \frac{(x_i - y_i)^2}{(\omega_i^{-1}(R))^2}. \tag{21}$$

But by Minkowski inequality

$$\left(\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{(\omega_i^{-1}(R)\,)^2}\right)^{1/2} \geq \left(\sum_{i=1}^{n}\frac{(x_i-x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} - \left(\sum_{i=1}^{n}\frac{(y_i-x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} \geq 8-1=7.$$

Besides $t-\tau \geq \frac{49}{4\beta S}$, then from (17) $\left(bR^2 \leq \frac{49}{4\beta S} \leq t-\tau\right)$ and monotonicity of $v(t)$ up to the first maximum, we deduce

$$\sup_{\substack{(x,t)\in s(C^5)\\(y,\tau)\in C^7}} G(x-y,t-\tau) \leq R^{-2S}\cdot b^S\cdot e^{-S}. \tag{22}$$

Now let's estimate $\inf\limits_{\substack{(x,t)\in s(C^6)\\(y,\tau)\in C^7}} G(x-y,t-\tau)$. Let $x\in\partial E_R^{x^0}(1)$, $y\in\partial E_R^{x^0}(1)$. Then

$$\left(\sum_{i=1}^{n}\frac{(x_i-y_i)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} \leq \left(\sum_{i=1}^{n}\frac{(x_i-x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} + \left(\sum_{i=1}^{n}\frac{(y_i-x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} \leq 1+1\leq 2,$$

$\frac{bR^2}{4} < t-\tau < bR^2$ and

$$\inf_{\substack{(x,t)\in s(C^6)\\(y,\tau)\in C^7}} G(x-y,t-\tau) \geq R^{-2S}\cdot e^{-s}\cdot e^{-\frac{16s}{49}}. \tag{23}$$

Let introduce an auxiliary function

$$W(x,t)=M\left[1-U(x,t)+R^{-2s}\cdot(be)^{-s}\cdot P_R(E_R)\right]-u(x,t), \text{ where } M=\sup_D u.$$

By Lemma 1, the function $W(x,t)$ - L-superelliptic-parabolic in D. According to the inequality (22) $W(x,t)\geq 0$ for $(x,t)\in\Gamma(D)\cap S(C^5)$. Moreover $W(x,t)\geq 0$ for $(x,t)\in\Gamma(D)\cap C^5$ by virtue of the inequality (19). Finally $W(x,t)\geq 0$ for $(x,t)\in F(C^5)$, $(x,t)\overline{\in}E_R$. Thus $W(x,t)\geq 0$ for $(x,t)\in\Gamma(D)$. By the maximum principle $W(x,t)\geq 0$ in D and in particular, given (23) and (20)

$$\sup_{D\cap C^6} u(x,t) \leq M\left[1-\inf_{D\cap C^6} U(x,t)+R^{-2S}\cdot(be)^{-S}\cdot P_R(E_R)\right]\leq$$

$$\leq M\left[1-(b)^{-S}\cdot\exp\left(-\frac{16S}{49}\right)\cdot(P_R(E_R)-\varepsilon)\cdot R^{-2S}+\right.$$
$$\left.+R^{-2S}\cdot(be)^{-S}\cdot P_R(E_R)\right]=$$
$$=M\left[1-b^{-S}\cdot\left(\exp\left(-\frac{16S}{49}\right)-\exp(-S)\right)\times\right.$$
$$\left.\times R^{-2S}\cdot P_R(E_R)+\varepsilon\cdot b^{-S}\cdot\exp\left(-\frac{16S}{49}\right)\cdot R^{-2S}\right]. \qquad (24)$$

Now allowing for that arbitrariness of ε, we arrive at the required inequality (18) from (24).

◀

Corollary 1. *If the conditions of Lemma 3 are fulfilled, and E_R contains a cylinder $C_{R:\rho}^{t'-\rho^2R^2,t'}(x')$, then $\sup\limits_{D} u(x,t)\geq(1+\eta_2(\gamma,n,\rho))\cdot\sup\limits_{D\cap C^6}u(x,t)$.*

Lemma 4. *Let the conditions of the previous Lemma 3 be fulfilled. Then there exists such $\delta(\gamma,n)$ that if $mesD\leq\delta\cdot mesC^5$ and $R\leq R_0$, then*

$$\sup_{D}u(x,t)\geq(1+\eta_3)\cdot\sup_{D\cap C^6}u(x,t),\ \textit{where} \qquad (25)$$
$$\eta_3=\frac{b}{2}.$$

Proof. Lets consider an auxiliary function

$$W_1(x,t)=M\left[1+\frac{b}{64}\cdot\sum_{i=1}^{n}\frac{(x_i-x_i^0)^2}{(\omega_i^{-1}(R))^2}+\frac{t^0-t}{R^2}-b\right]-u(x,t).$$

If is easy to see that

$$LW_1\leq\left(\frac{bn\cdot c_2}{32\gamma}+1\right)\frac{M}{R^2}=\frac{M_5\cdot c_5(\gamma,n,b)}{R^2}. \qquad (26)$$

On the other hand $W_1(x,t)\Big|_{\Gamma(D)}\geq 0$. Let's consider the domain $D_1\subset C^5$, $D\subset D_1$, $\tilde{C}_5=C_{R:8,5}^{t^0-bR^2,t^0}(x^0)$, $D'=D_1\cap\tilde{C}_5$, mes$D'\leq 2$mes$D$ and the function $\zeta(x,t)$ such that $\zeta(x,t)=1$ for $(x,t)\in D$, $\zeta(x,t)=0$ for $(x,t)\notin D'$.

Let $Z(x,t)$ be a solution of the following first boundary value problem

$$\begin{cases} LZ=-\dfrac{c_6}{R^2}\cdot\zeta(x,t), \quad (x,t)\in\tilde{C}^5;\\ Z\Big|_{\Gamma(\tilde{C}^5)}=0. \end{cases} \qquad (27)$$

By a maximum principle $Z(x,t) \geq 0$ for $(x,t) \in \tilde{C}^5$. Besides, by A.D. Alexandrov - N.V.Krylov inequality [11]

$$\sup_{\tilde{C}^5} Z(x,t) \leq \frac{c_7(n,\gamma) \cdot \left[\mathrm{mes} E_R^{x^0}(8,5)\right]^{1/(n+1)}}{\left(\inf\limits_{\tilde{C}^5} \det(a_{ij}(x,t))\right)^{1/(n+1)}} \cdot \left\| \frac{\zeta \cdot c_6}{R^2} \right\|_{L_{n+1}(\tilde{C}^5)} \leq$$
$$\leq c_8(n,\gamma) \cdot (2b\delta)^{1/(n+1)}. \tag{28}$$

Now if we put $W_2(x,t) = W_1(x,t) + M \cdot Z(x,t)$, then by virtue of (26) the function $W_2(x,t)$ is L-superelliptic-parabolic in D, $W_2(x,t)\Big|_{\Gamma(D)} \geq 0$ and by the maximum principle, allowing for (28)

$$\sup_{D \cap C^6} u(x,t) \leq M \left[1 - \frac{47b}{64} + c_8 \cdot (2b\delta)^{1/(n+1)}\right]. \tag{29}$$

Choose δ such that $c_8 \cdot (2b\delta)^{1/(n+1)} \leq \frac{15b}{64}$.

Then the inequality (25) follows from (29).

◀

Corollary 2. *Let the conditions of Corollary 1 be satisfied with respect to the region D and let in D define a positive L superelliptic-parabolic function $v(x,t)$ continuous in $\overline{D}$ and equal to unity on $\Gamma(D) \cap C^5$. Then, if $R \leq R_0$, then*

$$\inf_{D \cap C^6} v(x,t) \geq \eta_2'' = \frac{\eta_2'}{1+\eta_2'}, \tag{30}$$

where $\eta_2' = \frac{\eta_2}{2}$, $R_0 = R_0(\gamma, n)$.

Proof. Let

$$D' = \left\{(x,t) \,\middle|\, (x,t) \in D,\ v(x,t) < 1\right\}.$$

Consider the function $u(x,t) = 1 - v(x,t)$ for $(x,t) \in D'$. Applying Corollary 1 to the function $u(x,t)$, we get

$$1 - \inf_{D'} v \geq (1+\eta_2') \cdot \left(1 - \inf_{D' \cap C^6} v\right), \text{ i.e.}$$
$$\inf_{D' \cap C^6} v(x,t) \geq \frac{\eta_2}{1+\eta_2'}. \tag{31}$$

Now it is enough to remark that the required estimate of (30) is also proved.

◀

Analogously it is proved

Corollary 3. *Let the conditions of Lemma 4 be satisfied with respect to the region D and in D defined a positive L-superelleptic-parabolic function $v(x,t)$, continuous in $\overline{D}$ and turning to unity on $\Gamma(D)\cap C^5$. Then if $R \le R_0$, then*

$$\inf_{D\cap C^6} v(x,t) \ge \eta_3'' = \frac{\eta_3'}{1+\eta_3'}. \tag{32}$$

where $\eta_3' = \eta_3/2$ and $R_0 = R_0(\gamma,n)$.

Lemma 5. *Let a domain D be contained in the cylinder C^5 and have limit points on $\Gamma(C^5)$ and intersect C^6. Then let continuous in $\overline{D}$, and turning to unity on $\Gamma(D)\cap C^5$ positive L-supereliptic-parabolic function $v(x,t)$ be determined in D. Then if $E_R = C^7\setminus D$, $mesE_R \ge \sigma_1 \cdot mesC^7$, $\sigma_1 > 0$ and $R \le R_0$, then*

$$\inf_{D\cap C^6} v(x,t) \ge \eta_4(\gamma,n,\sigma_1). \tag{33}$$

Proof. Consider a cylinder

$$C^7(\rho_0) = C_{R:1-\rho_0}^{t^0-b(1-\rho_0)R^2,\,t^0-\frac{b}{2}(1+\rho_0)R^2}(x^0), \quad \rho_0 \in \left(0,\frac{1}{4}\right].$$

We choose and fix $\rho_0 > 0$ such that $\mathrm{mes}(C^7\setminus C^7(\rho_0)) = \frac{\sigma_1}{2}\cdot \mathrm{mes}C^7$.

It is clearly shown that ρ_0 depends only on n and σ_1. Let us denote by E^0 the set of interior intersection points of $E_R\cap C^7(\rho_0)$. Then

$$\begin{aligned}\mathrm{mes}E^0 \;\ge\; & \mathrm{mes}E_R - \mathrm{mes}(C^7\setminus C^7(\rho_0)) \ge \\ & \ge \sigma_1\cdot \mathrm{mes}C^7 - \frac{\sigma_1}{2}\cdot \mathrm{mes}C^7 = \frac{\sigma_1}{2}\cdot \mathrm{mes}C^7.\end{aligned} \tag{34}$$

Let (x',t') be an arbitrary point from E^0, and

$$\begin{aligned}& C_\nu^5(x',t') = C_{R:8\nu}^{t'-b(\nu R)^2,\,t'}(x'), C_\nu^6(x',t') = C_{R:\nu}^{t'-\frac{b}{4}(\nu R)^2,\,t'}(x'),\\ & C_\nu^7(x',t') = C_{R:\nu}^{t'-b(\nu R)^2,\,t'-\frac{b}{2}(\nu R)^2}(x')\end{aligned}$$

where $\nu\in(0,\rho_0]$ is such that $C_\nu^5(x',t')\in C^7$. We denote by $\nu(x',t')$ the exact upper edge of those $\nu\in(0,\rho_0]$ for which the following is true $\mathrm{mes}(D\cap C_\nu^5(x',t')) \le \delta\cdot \mathrm{mes}C_\nu^5(x',t')$, where δ is the constant of Lemma 4.

There are two possible cases:

1. $\nu(x',t') = \rho_0$,

2. $\nu(x',t') < \rho_0$.

Let case 1. take place. Then according to Corollary 3 either $D \cap C^6_{\rho_0}(x', t') = \varnothing$, either, $\inf\limits_{D \cap C^6_{\rho_0}(x',t')} v(x,t) \geq \eta''_3$.

Let $D' = \{(x,t) : (x,t) \in D, v(x,t) < \eta''_3\}$. From the above, it follows that $C^7 \setminus D'$ contains a cylinder $C^6_{\rho_0}(x', t')$. Applying to the function $\frac{v(x,t)}{\eta''_3}$ Corollary 3, we obtain $\inf\limits_{D' \cap C^6(x',t')} \frac{v(x,t)}{\eta''_3} \geq \eta''_2$, and the constant η''_2 only depends on γ, n and σ_1. Given that $v(x,t) \geq \eta'_3$ for $(x,t) \in D \setminus D'$, we conclude $\inf\limits_{D \cap C^6} v(x,t) \geq \eta''_2 \cdot \eta''_3$.

Thus, if case 1. holds, then Lemma 5 is proved.

Let case 2. take place. If for some other point $(x'', t'') \in E^0$ the case 1. is satisfied, then again Lemma 5 is proved.

So it remains to consider the situation when for any point $(x,t) \in E^0$ there is a case 2. Let's cover E^0 with cylinders $C^6_{\nu(x,t)}(x,t) \in E^0$ and choose from this coverage the countable under covered $\left\{C^6_{\nu_k}(x^k, t^k)\right\}, k = 1, 2, ...,$ of finite multiplicity $q_1(n)$ so that at that

$$\text{mes}\left[\bigcup_{k=1}^{\infty} (D \cap c^5_{\nu_k}(x^k, t^k))\right] \leq q_2(n) \text{mes}\left[\bigcup_{k=1}^{\infty} (D \cap c^6_{\nu_k}(x^k, t^k))\right].$$

By Corollary 3 for every natural k, either $D \cap C^6_{\nu_k}(x^k, t^k) = \varnothing$, either

$$\inf_{D \cap C^6_{\nu_k}(x^k,t^k)} v(x,t) \geq \eta''_3.$$

Moreover, in consideration of the (34)

$$\begin{aligned} \text{mes}\left[\bigcup_{k=1}^{\infty} (D \cap C^6_{\nu_k}(x^k, t^k))\right] &\geq \frac{1}{q_2} \text{mes}\left[\bigcup_{k=1}^{\infty} (D \cap C^5_{\nu_k}(x^k, t^k))\right] \geq \\ &\geq \frac{1}{q_1 q_2} \cdot \sum_{k=1}^{\infty} \text{mes}(D \cap C^5_{\nu_k}(x^k, t^k)) = \\ &= \frac{\delta}{q_1 q_2} \cdot \sum_{k=1}^{\infty} \text{mes} C^5_{\nu_k}(x^k, t^k) \geq \\ &= \frac{\delta}{q_1 q_2} \cdot \sum_{k=1}^{\infty} \text{mes} C^6_{\nu_k}(x^k, t^k) \geq \\ &\geq \frac{\delta}{q_1 q_2} \cdot \text{mes} E^0 \geq \frac{\delta \sigma_1}{2 q_1 q_2} \cdot \text{mes} E^7 \end{aligned} \quad (35)$$

We denote $\frac{v(x,t)}{\eta''_3}$ by $v_1(x,t)$ and let $D_1 = \{(x,t) : (x,t) \in D, v_1(x,t) < 1\}$.

Then if $E_R^1 = C^7 \setminus D_1$, then according to the condition on E_R and (35)

$$\text{mes}E_R^1 \geq \sigma_1 \cdot \text{mes}C^7 + \frac{\delta\sigma_1}{2q_1q_2} \cdot \text{mes}C^7 = \sigma_1 \cdot (1 + \frac{\delta}{2q_1q_2}) \cdot \text{mes}C^7.$$

Therefore, the estimate (33) is proved.

◀

Let's apply to the function $v_1(x,t)$ the same procedure as for the function $v(x,t)$. Then we can prove either Lemma 5 or we obtain that if $D_2 = \left\{(x,t) : (x,t) \in D,\ \frac{v_1(x,t)}{\eta_3''} < 1\right\}$, $E_R^2 = C^7 \setminus D_2$, then

$$\text{mes}E_R^2 \geq \sigma_1 \cdot \left(1 + \frac{\delta}{q_1q_2}\right)^2 \cdot \text{mes}C^7.$$

Let's continue the process following. Clearly, in this chain, alternative 2. cannot repeat more than $l_0(n,\delta,\sigma_1)$ times, where l_0 is the smallest natural number for which

$$\sigma_1 \cdot \left(1 + \frac{\delta}{2q_1q_2}\right)^{l_0} > 1.$$

Corollary 4. *Let in the cylinder C^5 be a arranged a domain D having limit points on $\Gamma(C^5)$ and intersecting C^6. Let $u(x,t)$ be a positive L-subelliptic-parabolic function, continuous in $\overline{D}$, and vanishing on $\Gamma(D) \cap C^5$. Then if $E_R = C^7 \setminus D$, mes$E_R \geq \sigma_1 \cdot$ mesC^7, $\sigma_1 > 0$, and $R \leq R_0$, then*

$$\sup_D u(x,t) \geq (1 + \eta_4) \cdot \sup_{D \cap C^6} u(x,t). \tag{36}$$

Proof. Let $v(x,t) = 1 - \frac{u(x,t)}{M}$, where $M = \sup_D u(x,t)$. Applying Lemma 5 to the function $v(x,t)$, we obtain that $1 - \frac{\sup_{D\cap C^6} u}{M} \geq \eta_4$. Hence the required estimate of (36) follows. Let $C^8 = C_{R:9}^{-bR^2, -\frac{3b}{4}R^2}(0)$.

◀

Lemma 6. *Let a domain D having limit points on $\Gamma(C^3)$ and intersecting C^4 be arranged on C^3. Then let a continuous in $\overline{D}$ and vanishing on $\Gamma(D) \cap C^3$ positive L - subelliptic-parabolic function $u(x,t)$ be determined on D. Then if the conditions (2)-(2) are fulfilled with respect to the coefficients of an operator L, and $H_R = C^8 \setminus D$, mes$H_R \geq \sigma \cdot$ mesC^8, $\sigma > 0$, $R \leq R_0$, then*

$$\sup_D u(x,t) \geq (1 + \eta(\gamma, n, \sigma)) \cdot \sup_{D \cap \Gamma(C^4)} u(x,t). \tag{37}$$

Proof. Without loss of generality, we may assume that $\sup\limits_{D\cap\Gamma(C^4)} u(x,t) = 1$.

Let $(x^*,t^*) \in D\cap\Gamma(C^4)$ be the point at which $u(x^*,t^*) = 1$. Suppose at the beginning that $(x^*,t^*) \in F(C^4)$, i.e., $(x^*,t^*) = (x^*,t^0)$, where $t^0 = -\frac{bR^2}{8}$. On $F(C^4)$, choose the minimum number of points $(x^1,t^0),\ldots,(x^m,t^0)$ such that

1. $\overline{C}_8 \subset \bigcup\limits_{i=1}^{m} C^7(x^i,t^0)$;
2. one of the points (x^i,t^0) coincides with the point (x^*,t^0);
3. for any $i, 1\le i\le m$, will be found $j, 1\le j\le m$ such that $x^j \in \partial E^{x^i}_{\frac{R}{A^m}}(1)$, where the constant $A(n) > 1$ will be selected later.

Clearly, the number m depends only on n. From the properties of the coating it follows, that for any $i_0, 1\le i_0\le m$, there is a chain $(x^{i_1},t^0),\ldots,(x^{i_k},t^0)$ such that $(x^{i_k},t^0) = (x^*,t^0)$,$x^{i_{e+1}} \in \partial E^{x^{i_l}}_{\frac{R}{A^m}}(1)$, $l = 0,1,\ldots,k-1$. From the condition on H_R we conclude the existence of $i_0, 1\le i_0\le m$ such that

$$\text{mes}\left(H_R\cap C^7(x^{i_0},t^0)\right) \ge \frac{\text{mes}H_R}{m} \ge \frac{\sigma}{m}\text{mes}C^8. \tag{38}$$

It is easy to see that $\text{mes}C^8 \ge \text{mes}C^7(x^{i_0},t^0)$. Therefore, from the (38) it follows that

$$\text{mes}\left(H_R\cap C^7(x^{i_0},t^0)\right) \ge \frac{\sigma}{m}\cdot\text{mes}C^7(x^{i_0},t^0). \tag{39}$$

Let $\delta_1 = \frac{\eta_4}{2(1+\eta_4)}$, where the constant is η_4 of Lemma 5 is taken at $\sigma_1 = \frac{\sigma}{m}$. Let's assume that $\sup\limits_{D\cap C^6(x^{i_0},t^0)} u(x,t) \ge 1-\delta_1$ .Then according to Corollary 4 and (39)

$$\begin{aligned}\sup_D u(x,t) &\ge (1+\eta_4)\sup_{D\cap C^6(x^{i_0},t^0)} u(x,t) \ge (1+\eta_4)\cdot(1-\delta_1) = \\ &= 1+\frac{\eta_4}{2} = (1+\frac{\eta_4}{2})\cdot\sup_{D\cap\Gamma(C^4)} u(x,t),\end{aligned}$$

and in this case the statement of Lemma 6 is proved.

Now, let's say $u(x,t) < 1-\delta_1, (x,t)\in D\cap C^6(x^{i_0},t^0)$.

Consider the function $v_1(x,t) = u(x,t)-1+\delta_1$. It is not difficult to see that the function $v_1(x,t)$ is L- subelliptic-parabolic in D, since $\delta_1 < 1$.

Let $D_1 = \{(x,t) : (x,t)\in D, v_1(x,t) > 0\}$. The assumption is that the cylinder $C^6(x^{i_0},t^0)$ located in addition to D_1.

For $(x',t')\in\Gamma(C^4)$ denote by $C^i_{R'}(x',t')$ cylinder $C^i(x',t')$, $i=5,6,7$ highlighting that in it $R=R'$. Let's find one now $A>1$, that $C^5_R(x',t')\subset C^6_{AR}(x',t')$. Clearly, it is enough for the inclusion to be fair that

$$bR^2\le b\cdot\frac{(AR)^2}{4},\ 8\omega_i^{-1}(R)\le\omega_i^{-1}(AR),\ i=1,\dots,n.$$

The last inequalities are satisfied if we fix $A=\max\{2,\alpha_1\}$.

Now, let's say $(x^{i_1},t^0),\dots,(x^{i_k},t^0)$ above-mentioned chain. By construction $C^7_{\frac{R}{A}}(x^{i_1},t^0)\setminus D_1$ contains a cylinder $C^{t'-b(\frac{R}{A}\cdot\rho_1)^2,t'}_{\frac{R}{A}\cdot\rho_1}(x')$, where ρ_1 depends only on n.

Let's assume that $\sigma_0=\frac{\eta_2}{2(1+\eta_2)}$, where the constant η_2 of Corollary 1 is taken at $\rho=\rho_1$.

Let's assume that

$$\sup_{D_1\cap C^6_{\frac{R}{A}}(x^{i_1},t^0)} v_1(x,t)\ge\delta_1(1-\sigma_0),\text{ i.e.}$$

$$\sup_{D_1\cap C^6_{\frac{R}{A}}(x^{i_1},t^0)} u(x,t)\ge 1-\delta_1\sigma_0.$$

Using Corollary 1, we obtain

$$\begin{aligned}\sup_{D_1\cap C^6_{R}(x^{i_1},t^0)} v_1(x,t)\ &\ge\ (1+\eta_2)\sup_{D_1\cap C^6_{\frac{R}{A}}(x^{i_1},t^0)} v_1(x,t)\ge\\ &\ge(1+\eta_2)\cdot\delta_1\cdot(1-\sigma_0).\end{aligned}$$

Thus

$$\begin{aligned}\sup_D u(x,t)\ &\ge\ \sup_{D_1\cap C^6_{R}(x^{i_1},t^0)} u(x,t)\ge(1-\delta_1)+\\ &+(1+\eta_2)\cdot\delta_1\cdot(1-\sigma_0)=1+\frac{\delta_1\eta_2}{2}=\\ &=\left(1+\frac{\delta_1\eta_2}{2}\right)\cdot\sup_{D\cap\Gamma(C^4)} u(x,t),\end{aligned}$$

and in this case the statement of Lemma 6 is proved.

Let $u(x,t)<1-\delta_1\sigma_0$, $(x,t)\in D_1\cap C^6_{\frac{R}{A}}(x^{i_1},t^0)$, then consider L-subelliptic-parabolic in D function $v_2(x,t)=u(x,t)-(1-\delta_1\sigma_0)$. Let $D_2=\left\{(x,t)\middle|(x,t)\in D, v_2>0\right\}$. The assumption is that the cylinder $C^6_{\frac{R}{A}}(x^{i_1},t^0)$ is located in addition to D_2. If now

$$\sup_{D_1\cap C^6_{\frac{R}{A^2}}(x^{i_2},t^0)} v_2(x,t)\ge\delta_1\sigma_0(1-\sigma_0),\text{ i.e.}$$

$$\sup_{D_1\cap C^6_{\frac{R}{A^2}}(x^{i_2},t^0)} u(x,t)\geq 1-\delta_1\sigma_0^2,$$

then, applying Corollary 1, we obtain

$$\sup_D u(x,t) \geq \sup_{D\cap C^6_{\frac{R}{A^2}}(x^{i_2},t^0)} u(x,t)\geq 1-\delta_1\sigma_0+$$
$$+(1+\eta_2)\delta_1\sigma_0(1-\sigma_0)=1+\frac{\delta_1\sigma_0\eta_2}{2}=$$
$$=\left(1+\frac{\delta_1\sigma_0\eta_2}{2}\right)\cdot\sup_{D\cap\Gamma(C^4)} u(x,t),$$

and in this case the statement of Lemma 6 is proved.

If, however $u(x,t)<1-\delta_1\sigma_0^2$, $(x,t)\in D\cap C^6_{\frac{R}{A^2}}(x^{i_2},t^0)$, then we will continue the process as follows. No later than the k-th step, we prove Lemma 6, since $u(x^{i_k},t^0)=u(x^*,t^0)=1$. So Lemma 6 is proved if $(x^*,t^*)\in\overline{F}(C^4)$.

Now, let's say $(x^*,t^*)\in S(C^4)$ and $t^*>t^0$. Clearly $x^*\in\partial E^0_R(9)$. From the above considerations it follows that either Lemma 6 is proved, either $u(x,t)<1-\delta_1\sigma_0^m$ for $(x,t)\in D\cap C^6_{\frac{R}{A^m}}(x^*,t^0)$.

Let's choose on the segment I, connecting the points (x^*,t^0) and (x^*,t^*), minimum number of points $(x^*,t^1),\ldots,(x^*,t^p)$ so that

4. $I\subset\bigcup\limits_{i=1}^{p}\overline{C}^6_{\frac{R}{A^m}}(x^*,t^i)$; $t^p=t^*$;

5. at the intersection of $\overline{C}^6_{\frac{R}{A^m}}(x^*,t^i)\cap\overline{C}^6_{\frac{R}{A^m}}(x^*,t^{i+1})$ cylinder is contained $C^7_{\frac{R}{A^m}}(x^*,t^{i+1})$,$i=0,1,\ldots,p-1$.

Clearly, p depends only on n. Consider the function $W_1(x,t)=u(x,t)-1+\delta_1\cdot\sigma_2^m$, where $\sigma_2=\min\left\{\sigma_0,\frac{\overline{\eta}_2}{2(1+\overline{\eta}_2)}\right\}$,$\overline{\eta}_2$-constant η_2 of Corollary 1, taken at $\rho=A^{-1}$.

Let $D^1=\left\{(x,t)\middle|(x,t)\in D, W_1(x,t)>0\right\}$. By assumption, the cylinder $C^6_{\frac{R}{A^m}}(x^*,t^0)$ is located in the supplement of D^1. If

$$\sup_{D^1\cap C^6_{\frac{R}{A^m}}(x^*,t^1)} W_1(x,t)\geq\delta_1\cdot\sigma_2^m\cdot(1-\sigma_2),\text{ i.e.}$$
$$\sup_{D^1\cap C^6_{\frac{R}{A^m}}(x^*,t^1)} u(x,t)\geq 1-\delta_1\cdot\sigma_2^{m+1},$$

then applying Corollary 1, we obtain

$$\sup_{D^1\cap C^6_{\frac{R}{A^{m-1}}}(x^*,t^1)} W_1(x,t)\geq(1+\overline{\eta_2})\cdot\delta_1\cdot\sigma_2^m\cdot(1-\sigma_2).$$

Thus

$$\sup_D u(x,t) \geq \sup_{D^1\cap C^6_{\frac{R}{A^{m-1}}}(x^*,t^1)} u(x,t) \geq 1-\delta_1\cdot\sigma_2^m + $$
$$+(1+\overline{\eta}_2)\cdot\delta_1\cdot\sigma_2^m\cdot(1-\sigma_2) \geq 1+\frac{\delta_1\sigma_2^m\cdot\overline{\eta}_2}{2} =$$
$$= \left(1+\frac{\delta_1\sigma_2^m\cdot\overline{\eta}_2}{2}\right)\sup_{D\cap\Gamma(C^4)} u(x,t),$$

and in this case the statement of Lemma 6 is proved. If $u(x,t) < 1-\delta_1\sigma_2^{m+1}$, $(x,t)\in D^1\cap C^6_{\frac{R}{A^m}}(x^*,t^1)$, then we continue the process as follows. At the latest p-step, we prove Lemma 6, since $u(x^*,t^p) = u(x^*,t^*) = 1$.

The estimate (37) is completely proved.

◀

Theorem 1. *Let a domain D have limit points on $\Gamma(C^3)$ and intersect C^4 be arranged on C^3. Then let a continuous in $\overline{D}$ and vanishing on $\Gamma(D)\cap C^3$ positive L-subelliptic-parabolic function $u(x,t)$ be determined on D. Then if $mesH_R \geq \sigma\cdot mesC^8$, $\sigma>0$, and $R\leq R_0$, then $\sup_D u(x,t) \geq (1+\eta)\cdot\sup_{D\cap C^4} u(x,t)$.*

We denote by $C_\lambda(D)$, $0<\lambda<1$ the Banach space of functions $u(x,t)$ defined on D, with finite norm

$$\|u\|_{C_\lambda(D)} = \sup_D|u| + \sup_{\substack{(x,t),(y,\tau)\in D\\(x,t)\neq(y,\tau)}} \frac{|u(x,t)-u(y,\tau)|}{\left(|x-y|+\sqrt{|t-\tau|}\right)^\lambda}.$$

Let's $\operatorname{osc}_{Q_T} u(x,t) = \sup_{Q_T} u(x,t) - \inf_{Q_T} u(x,t)$, $\rho>0$, $C_\rho(x,t) = C_{\rho:9}^{t-\frac{bR^2}{8},t}(x)$, $\overline{C^1}\subset D$, $D^\rho = \{(x,t)|(x,t)\in D, C_\rho(x,t)\subset D\}$.

Theorem 2. *Let in domain $D\subset R_{n+1}$ a solution $u(x,t)$ of the equation (1) be determined, moreover as to the coefficients of the operator L, the conditions (2)-(4) be fulfilled, $n\geq 1$ and $(0,0)\in D$. Then if $R\leq R_0$ is such that $\overline{C^1}\subset D$. Then*

$$\operatorname{osc}_{C^1} u(x,t) \geq \left(1+\frac{\eta}{2}\right)\operatorname{osc}_{C^4} u(x,t), \tag{40}$$

where η is the constant of Theorem 1 at $\sigma=\frac{1}{2}$.

Proof. In the proof suggested below, only Theorem 1 will be used. In this plan, if $M_1 = \sup_{C^1} u(x,t)$, $m_1 = \inf_{C^1} u(x,t)$,$M_2 = \sup_{C^4} u(x,t)$, $m_2 = \inf_{C^4} u(x,t)$ and the inequality of the form (40) is valid for the function $v(x,t) = u(x,t) - \frac{M_2+m_2}{2}$, then it is fulfilled for the function $u(x,t)$. But

$$\sup_{C^4} v(x,t) = \sup_{C^4} u - \frac{M_2+m_2}{2} = M_2 - \frac{M_2+m_2}{2} = \frac{M_2-m_2}{2},$$

$$\inf_{C^4} v(x,t) = \inf_{C^4} u - \frac{M_2+m_2}{2} = m_2 - \frac{M_2+m_2}{2} = -\frac{M_2-m_2}{2},$$

Furthermore, it can always be considered that $M_2 - m_2 > 0$, otherwise inequality (40) is obvious. Therefore, without loss of generality, we will supposed that the $M_2 = 1, m_2 = -1$, i.e. $\operatorname{osc}_{C^4} u(x,t) = 2$. Let $D^+ = \{(x,t)|(x,t) \in C^1, u(x,t) \geq 0\}$, $D^- = \{(x,t)|(x,t) \in C^1, u(x,t) < 0\}$.Obviously, both of these sets are not empty.

At least one of the following inequalities is satisfied:

1. $\operatorname{mes}(C^8 \setminus D^+) \geq \frac{1}{2}\operatorname{mes} C^8$,

2. $\operatorname{mes}(C^8 \setminus D^-) \geq \frac{1}{2}\operatorname{mes} C^8$.

Let the case 1. take place for definiteness. We note that alternative 2. reduces to 1. multiplying the solution $u(x,t)$ by (-1). Let's denote by D' that connected component of the set D^+ which contains the point $(x^0, t^0) \in \Gamma(C^4)$, where $u(x^0,t^0) = 1$. By applying Theorem 1 to the function $u(x,t)$ in D' with the constant $\sigma = \frac{1}{2}$, we obtain that

$$M_1 \geq (1+\eta)M_2 = 1+\eta, \text{ i.e.}$$
$$M_1 - m_1 \geq 1 + \eta - m_1 \geq 1 + \eta - m_2 = 2 + \eta =$$
$$= 2\left(1 + \frac{\eta}{2}\right) = \left(1 + \frac{\eta}{2}\right) \cdot (M_2 - m_2),$$

and the required estimate of (40) is proved.

◀

Corollary 5. *Let* $C_R(x^0,t^0) = C_{R:9}^{t^0 - \frac{bR^2}{8}, t^0}(x^0)$, $\nu = \max\left\{3, \frac{\alpha_2}{\alpha_1}\right\}$. *Then, if the conditions of Theorem 1 are satisfied and* $\overline{C}_{\nu R}(0,0) \in D$, *then at* $R \leq R_0$

$$\underset{C_{\nu R}(0,0)}{osc} u(x,t) \geq \left(1 + \frac{\eta}{2}\right) \underset{C_R(0,0)}{osc} u(x,t). \tag{41}$$

For the proof it is enough to notice that at the chosen ν there is an inclusion of $C^1 \subset C_{\nu R}(0,0)$.

Corollary 6. *An estimate of the form (41) is also valid in cylinders* $C_{\nu R}(x^0,t^0)$ *and* $C_R(x^0,t^0)$ *respectively, unless* $R \le R_0$ *and* $C_{\nu R}(x^0,t^0) \subset (D \cup \gamma(D))$. *Here* $\gamma(D)$ *is the top cover of the region* D.

Theorem 3. *Let* $u(x,t)$ *of the equation (1) be defined in the region* D, *whose coefficients satisfy the conditions (2)-(4). Then for any* $\rho > 0$ *there exist constants* $\lambda(\gamma,n)$, $C_9(\gamma,n,\rho)$ *such that*

$$\|u\|_{C_\lambda(D^\rho)} \le C_9 \cdot \|u\|_{C(D)}. \tag{42}$$

Proof. Let (x^1,t^1) and (x^2,t^2) be two arbitrary points from the D^ρ. For definiteness we will assume that $t^2 < t^1$. Let us fix an arbitrary small enough $\rho > 0$. Two cases are possible:

1. $(x^2,t^2) \in C_\rho(x^1,t^1)$,
2. $(x^2,t^2) \notin C_\rho(x^1,t^1)$.

Let us first consider case 1. Let for $m = 0,1,2,\dots$ $C(m)$ denote the cylinder $C_{\rho\nu^{-m}}(x^1,t^1)$. Clearly, there exists a non-negative integer m_0 for which

$$(x^2,t^2) \in C(m_0), \quad (x^2,t^2) \notin C(m_0+1). \tag{43}$$

It follows from (43) that either $x^2 \notin E^{x^1}_{\rho\nu^{-m_0-1}}(9)$, either $t^1 - t^2 \ge \frac{b\rho^2\nu^{-2m_0-2}}{8}$.

If $x^2 \notin E^{x^1}_{\rho\nu^{-m_0-1}}(9)$, then $\sum\limits_{i=1}^{n} \frac{(x_i^1-x_i^2)^2}{(\omega_i^{-1}(\rho\nu^{-m_0-1}))^2} \ge 81$. Thus there you'll find $i_0, 1 \le i_0 \le n$ such that $|x^1_{i_0} - x^2_{i_0}| \ge \frac{9}{\sqrt{n}} \cdot \omega_{i_0}^{-1}(\rho\nu^{-m_0-1})$. So, in any case

$$\begin{aligned} |x_1 - x_2| + \sqrt{t^1 - t^2} \ \ \ge\ \ & \frac{9}{\sqrt{n}} \cdot \omega_i^{-1}(\rho\nu^{-m_0-1}) + \\ & + \sqrt{\frac{b}{8}}\rho\nu^{-m_0-1} \ge c_{10}(\gamma,n)\omega_i^{-1}(\rho\nu^{-m_0-1}). \end{aligned} \tag{44}$$

Applying now successively Theorem 2 to the function $u(x,t)$ in cylinders $c(i)$ and $c(i+1)$,$i = 0,1,\dots,m_0-1$, we obtain

$$\operatorname*{osc}_{C(0)} u(x,t) \ge \left(1 + \frac{\eta}{2}\right)^{m_0} \cdot \operatorname*{osc}_{C(m_0)} u(x,t), \text{ i.e.}$$

$$\operatorname*{osc}_{C(m_0)} u(x,t) \le \frac{1}{\left(1+\frac{\eta}{2}\right)^{m_0}} \cdot \operatorname*{osc}_{C(0)} u(x,t) \le$$

$$\le \frac{1+\frac{\eta}{2}}{\left(1+\frac{\eta}{2}\right)^{m_0+1}} \cdot \operatorname*{osc}_{D} u(x,t) \le \frac{2\left(1+\frac{\eta}{2}\right)}{\left(1+\frac{\eta}{2}\right)^{m_0+1}} \cdot \|u\|_{C(D)}.$$

Thus, taking into account (43), we conclude that

$$\left|u(x^1,t^1)-u(x^2,t^2)\right| \le \frac{c_{11}(\gamma,n)}{\left(1+\frac{\eta}{2}\right)^{m_0+1}}\cdot\|u\|_{C(D)}. \tag{45}$$

Let us denote $p = 1+\frac{\eta}{2}$, then $p^{m_0+1} = \nu^{\log_\nu p^{m_0+1}} = \nu^{(m_0+1)\log_\nu p} = \nu^{(m_0+1)\lambda}$, where $\lambda = \log_\nu p$. Taking into account (44), we obtain

$$p^{-(m_0+1)} = \nu^{-(m_0+1)\lambda} = \frac{\left[c_{10}\cdot\omega_i^{-1}(\rho\cdot\nu^{-m_0-1})\right]^\lambda}{\nu^{(m_0+1)\lambda}\cdot\left[c_{10}\cdot\omega_i^{-1}(\rho\nu^{-m_0-1})\right]^\lambda} \le$$
$$\le \frac{\left[|x^1-x^2|+\sqrt{t^1-t^2}\right]^\lambda}{\left[c_{2.2}\cdot\nu^{m_0+1}\cdot\omega_i^{-1}(\rho\nu^{-m_0-1})\right]^\lambda}.$$

Using the latter estimate in (45), we arrive at the inequality

$$\left|u(x^1,t^1)-u(x^2,t^2)\right| \le \frac{c_{11}\cdot\left[|x^1-x^2|+\sqrt{|t^1-t^2|}\right]^\lambda\cdot\|u\|_{C(D)}}{\left[c_{10}\cdot\nu^{m_0+1}\cdot\omega_i^{-1}(\rho\nu^{-m_0-1})\right]^\lambda}. \tag{46}$$

Let alternative 2. now take place. Then either $x^2 \notin E_\rho^{x^1}(9)$, either $t^1-t^2 \ge \frac{b\rho^2}{8}$. Proceeding in the same way as in deriving equation (44), we obtain that in any case

$$|x^1-x^2|+\sqrt{t^1-t^2} \ge c_{12}(\gamma,n)\cdot\omega_i^{-1}(\rho). \tag{47}$$

Therefore, taking into account (47), we have

$$\left|u(x^1,t^1)-u(x^2,t^2)\right| \le 2\|u\|_{C(D)} \le$$
$$\le \frac{2(|x^1-x^2|+\sqrt{t^1-t^2})^\lambda}{(c_{12}\cdot\omega_i^{-1}(\rho))^\lambda}\cdot\|u\|_{C(D)}. \tag{48}$$

From (46) an (48) we conclude

$$\frac{|u(x^1,t^1)-u(x^2,t^2)|}{(|x^1-x^2|+\sqrt{t^1-t^2})^\lambda} \le c_{13}\cdot\|u\|_{C(D)}, \tag{49}$$

where $c_{13} = \max\left\{\frac{c_{11}}{\left[c_{10}\cdot\nu^{m_0+1}\cdot\omega_i^{-1}(\rho\nu^{-m_0-1})\right]^\lambda}, \frac{2}{\left[c_{12}\cdot\omega_i^{-1}(\rho)\right]^\lambda}\right\}$. Now it is enough to consider the arbitrariness of points (x^1,t^1) and (x^2,t^2) from D^ρ, and from (49) follows the required estimate (42) with $c_9 = c_{13}+1$.

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References

[1] Amanova, N.R. (2017). On strong solvability of the Dirichlet problem for a class of uniformly degenerate elliptic equations of second order. *Caspian Journal of Applied Mathematics, Ecology and Economics* **52** 3–25.

[2] Amanova, N.R. (2023). Weak solvability of the first boundary value problem for non-uniformly and strongly degenerate second-order elliptic–parabolic equations in divergent form. *Transactions of NAS of Azerbaijan, Series of Physical-Technical and Mathematical Sciences, Mathematics* **43**(4), 54–64.

[3] Amanova, N. R. and Amanov, R. A. (2025). Behavior near the boundary of the solutions of degenerate second-order elliptic–parabolic equations. *Journal of Contemporary Applied Mathematics* **9**(1), 1–9.

[4] Amanova, N.R. (2024). Some properties of solutions to the first boundary value problem for second-order elliptic–parabolic equations with discontinuous coefficients. *Proceedings of IAM*, Baku, **13**(2), 220–247.

[5] Amanov, R. A. and Ismailov, H. I. (2023). On equations of the form $\Delta u - \frac{\partial u}{\partial t} = f(x, t, u, Du)$. *Azerbaijan Journal of Mathematics* **13**(2), 137–158.

[6] Amanova, N.R. (2025). On quasilinear parabolic equations of higher order. *Azerbaijan Journal of Mathematics* **15**(1), 191–209.

[7] Amanov, R.A. and Amanova, N.R. (2025). On solvability of some quasilinear parabolic equations of higher order. *Azerbaijan Journal of Mathematics* **15**(2), 240–263.

[8] Amanova, N.R. and Khalilov, V.S. (2024). On the criterion of strong solutions of higher-order quasilinear equations. *Uzbek Mathematical Journal* **68**(1), 11–18.

[9] Amanov, R.S., Khalilov, V.S., and Amanova, N.V. (2025). Existence of strong solutions to semilinear elliptic systems of second order. *Kazakh Mathematical Journal* **25**(3), 39–59.

[10] Glagoleva, R.L. (1968). A priori estimates of Hölder norms and Harnack inequalities for solutions of linear second-order parabolic equations with discontinuous coefficients. *Matematicheskii Sbornik* **118**(2), 167–185.

[11] Krylov, N.V. and Safonov, M.V. (1979). An estimate of the probability of diffusion processes hitting a set of positive measure. *Doklady Akademii Nauk SSSR* **245**(1), 18–20.

[12] Krylov, N.V. and Safonov, M.V. (1980). Some properties of solutions of parabolic equations with measurable coefficients. *Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya* **44**(1), 161–175.

[13] Khalouta, A., & Efendiev, R. F. (2025). New methodology for a system of nonlinear time-fractional partial differential equations based on the Khalouta-Caputo-Katugampola fractional derivative. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics*, **45**(4), 105-122.

[14] Ladyzhenskaya, O.A. and Uraltseva, N. N. (1981). On estimates of Hölder constants for solutions of quasilinear parabolic equations in non-divergence form. *Uspekhi Matematicheskikh Nauk* **36**(4), 20–38.

[15] Mamedov, I.T. (1983). A theorem on the oscillation of solutions of second-order parabolic equations with discontinuous coefficients. *Izvestiya AN AzSSR, Series of Physical-Technical and Mathematical Sciences* **3**, 10–19.

[16] Mamedov, I.T. and Mushtagov, F.M. (1999). Harnack inequality for solutions of non-uniformly degenerate second-order parabolic equations. *Transactions of the Academy of Sciences of Azerbaijan, Series of Physical-Technical and Mathematical Sciences* **19**(1–2), 122–132.

[17] Ladyzhenskaya, O.A., Solonnikov, V.A., and Uraltseva, N.N. (1967). *Linear and Quasilinear Equations of Parabolic Type*. Nauka, Moscow, 736 pp.

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