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Strong Solvability of One Nonlocal Problem by the Spectral Method in Orlicz-Sobolev Spaces

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Abstract. In this work we consider a nonlocal problem for the Laplace equation on an unbounded domain and define the concept of a strong solution of this problem in Sobolev spaces generated by the Orlicz norm. We take advantage of the fact that the system of root functions of the spectral problem corresponding to this problem forms a basis for the Orlicz space. We apply the spectral method using Boyd indices in symmetric spaces and demonstrate the correct solvability of the problem in Orlicz-Sobolev spaces. Solving the problem in Orlicz-Sobolev spaces will pave the way for generalizing the solution to symmetric Sobolev spaces which have a more general structure.

Key Words and Phrases: Orlicz-Sobolev spaces, Laplace equation, nonlocal problem, strong solution

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1. Introduction

The classical theory of the solvability of linear elliptic equations has been extensively studied in the classical, strong, or weak sense (see e.g. [1, 2, 3, 4, 5, 6, 7]). However, some mechanical and mathematical physics problems do not fit to this theory. In [8] Moiseev studied such a problem that involved the following degenerate elliptic equation

$$y^m u_{xx} + u_{yy} = 0, \quad (x, y) \in (0, 2\pi) \times (0, +\infty), \tag{1}$$

$$u(x, 0) = f(x), \quad x \in (0, 2\pi), \tag{2}$$

$$u(0, y) = u(2\pi, y), \quad u_x(0, y) = 0, \quad y \in (0, +\infty), \tag{3}$$

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with $m \geq -2$. This is a nonlocal problem with semi-infinite lines as support boundaries, one of which is the normal derivative. Such problems have different characteristics compared to problems with local conditions. Frankl considered a mixed type equation with a non-local boundary condition in [9, 10]. Bitsadze and Samarskii considered a nonlocal problem for elliptic equations with supports on a portion of the domain's boundary, where these supports are independent of the other boundary conditions [11]. Ionkin and Moiseev [12] solved the boundary value problem for multi-dimensional parabolic equations with nonlocal conditions, where the supports are the characteristics and the improper parts of the boundary of the domain. Lerner [13] also addressed the problem (1) in the classical formulation.

In this article we will consider at problem (1)-(3) in the case $m = 0$, *i.e.* for the Laplace equation in an Orlicz-Sobolev space. We will define the notion of a strong solution of this problem and also prove the correct solvability of this problem using the Fourier method. When studying the solvability of elliptic equations in Orlicz-Sobolev spaces, it should be noted that these spaces exhibit some specific differences from other spaces due to their symmetric structure. It is worth noting that there are very few studies on this topic [14, 15].

When solving this problem in the classical sense, Moiseev [8] used the spectral method, based on the fact that the corresponding root functions

$$\{1;\, \cos nx;\, x \sin nx\}_{n\in\mathbb{N}}\,, \tag{4}$$

forms a Riesz basis in $L_2(0, 2\pi)$. Then the authors of the works [16, 17, 18, 19, 20], established the basisness of the system (4) in weighted Lebesgue and weighted grand Lebesgue spaces and using these facts, solved the problem (1) (in strong and weak sense) in corresponding Sobolev spaces generated by norm of these spaces. In addition, Bilalov et al. [21] investigated the basicity of the system in Orlicz spaces. Building on this fact, we will examine the strong solution in the corresponding Orlicz-Sobolev space.

2. Auxiliary Facts

2.1. Notations

First, we introduce some standard notations. $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ and δ_{ij} is the Kronecker delta symbol. $I = (0, 2\pi)$. $C_0^\infty(I)$ is the set of all infinitely differentiable functions on I with compact support in I. By $[X; Y]$ we will denote the Banach space of bounded linear operators acting from space X to Y; $[X] \equiv [X;X]$. $\alpha = (\alpha_1;\alpha_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ will be multiindex and $\partial^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$, where $|\alpha| = \alpha_1 + \alpha_2$. $L[M]$ denotes the linear span of the set M. X^* is the dual space of X and c denotes constant (maybe difference in various places). $f|_M$ denotes the restriction of f on M. p' is conjugate to number p : $\frac{1}{p} + \frac{1}{p'} = 1$.

2.2. Basis Properties

For the sake of completeness of the presentation, we give some necessary information from basis theory.

Definition 1. *Let X be a Banach space on field K. The system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a basis in the space X if for $\forall x \in X$, there is a unique sequence $\{a_n\}_{n\in N} \subset K$ such that*

$$x = \sum_{n=1}^{\infty} a_n x_n.$$

Definition 2. *If the condition*

$$x_n^*(x_k) = \delta_{nk}, \quad \forall n;k \in \mathbb{N},$$

is satisfied, the systems $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{x_n^*\}_{n\in\mathbb{N}} \subset X^*$ are said to be biorthogonal.

Definition 3 (Completeness)**.** *Let X be a Banach space. The system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is complete in X, if*

$$\overline{L[\{x_n\}_{n\in\mathbb{N}}]} = X.$$

The completeness criterion for a system in Banach spaces is given below.

Proposition 1 (Completeness Criterion)**.** *Let X be a Banach space. The system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is complete in $X \Leftrightarrow \varphi \in X^*$: $\varphi(x_n) = 0$, $\forall n \in \mathbb{N} \Rightarrow \varphi = 0$.*

Definition 4 (Minimality)**.** *The system $\{x_n\}_{n\in\mathbb{N}} \subset X$ is called minimal in X, if*

$$x_k \notin \overline{L[\{x_n\}_{n\in\mathbb{N}_k}]}, \quad \forall k \in \mathbb{N},\ \mathbb{N}_k = \mathbb{N} \setminus \{k\}.$$

The minimality criterion for a system in Banach spaces is given below.

Proposition 2 (Minimality Criterion)**.** *A necessary and sufficient condition for a system to be minimal in Banach space is that the system has a biorthogonal system.*

Proposition 3 (Basicity Criterion)**.** *The system $\{x_n\}_{n\in\mathbb{N}}$ forms a basis in the Banach space X if and only if, the following statements are true:*

1. *$\{x_n\}_{n\in\mathbb{N}}$ is complete in X;*

2. *$\{x_n\}_{n\in\mathbb{N}}$ is minimal in X with the biorthogonal system $\{x_n^*\}_{n\in\mathbb{N}} \subset X^*$;*

3. *The projectors $P_k(x) = \sum_{m=1}^{k} x_m^*(x)x_m$ are uniformly bounded $(\forall k \in \mathbb{N})$, i.e., there exists $C > 0$ such that*

$$\|P_k(x)\|_X \le C\|x\|_X, \quad \forall x \in X, \ \forall k \in \mathbb{N}.$$

2.3. Orlicz Spaces

Let us recall the necessary concepts and facts concerning Orlicz space.

Definition 5. *A continuous convex function $M : \mathbb{R} \to \mathbb{R}$ is called N-function if it is even and satisfies the conditions*

$$\lim_{u\to 0} \frac{M(u)}{u} = 0; \quad \lim_{u\to\infty} \frac{M(u)}{u} = \infty \ .$$

Definition 6. *Let M be an N-function. The function defined by*

$$M^*(v) = \max_{u\ge 0}[u(v) - M(u)],$$

is called N-function complement to M.

The function M^* can be defined as follows. Let $p : \mathbb{R}_+ \to \mathbb{R}_+ = [0; +\infty)$, be right continuous for $t \ge 0$, non-decreasing function that satisfies the condition $p(0) = 0, \ p(\infty) = \lim_{t\to\infty} p(t) = \infty$. Let us define

$$q(s) = \sup_{p(t)\le s} t \ , \qquad s \ge 0.$$

The function q has the same properties as the function p. In fact for $s > 0$ it is positive, for $s \ge 0$ it is right continuous, non-decreasing and meets the conditions

$$p(0) = 0, \ p(\infty) = \lim_{t\to\infty} p(t) = \infty.$$

M and M^* are complement to each other and these N-functions can be represented as follows

$$M(u) = \int_0^{|u|} p(t)dt, \quad M^*(v) = \int_0^{|v|} q(s)ds.$$

Definition 7. *The N-function M satisfies the following condition called Δ_2-condition for large values of u, if $\exists k > 0$ and $\exists u_0 \geq 0$:*

$$M(2u) \leq kM(u), \qquad \forall u \geq u_0.$$

Δ_2-condition is equivalent to fact that, for $\forall l > 1,\ \exists k(l) > 0$ and $\exists u_0 \geq 0$:

$$M(lu) \leq k(l)M(u), \qquad \forall u \geq u_0.$$

Now we can define the Orlicz space. Let M be some N-function, $\Omega \subset \mathbb{R}^n$ be a (Lebesgue) measurable set with finite measure. Denote by $L_0(\Omega)$ the set of all measurable functions in Ω. Let

$$\rho_M(u) = \int_\Omega M[u(x)]\, dx,$$

and

$$\widetilde{L}_M(\Omega) = \{u \in L_0(\Omega) : \ \rho_M(u) < +\infty\}.$$

$\widetilde{L}_M(\Omega)$ is called an Orlicz class. Let M and M^* be complement for each other N-functions. Assume

$$L_M(\Omega) = \{u \in L_0(\Omega) : \ |(u;v)| < +\infty,\ \forall v \in \widetilde{L}_{M^*}(\Omega)\},$$

here

$$(u;v) = \int_\Omega u(x)\overline{v(x)}\, dx.$$

$L_M(\Omega)$ is called Orlicz space. According to the norm $\|.\|_M$:

$$\|u\|_M = \sup_{\rho_{M^*}(v) \leq 1} |(u;v)|,$$

$L_M(\Omega)$ is a Banach space. In $L_M(\Omega)$ an equivalent norm to $\|.\|_M$ can be defined as

$$\|u\|_{(M)} = \inf\left\{\lambda > 0 : \rho_M\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

also known as the Luxemburg norm.

Proposition 4. $L_M(\Omega) = \widetilde{L}_M(\Omega)$ *and the closure of the set of bounded (including continuous) functions coincides with* $L_M(\Omega)$*, if N-function M satisfies the Δ_2-condition.*

For further details on these and related results, see monographs [22, 23].

Definition 8. *We will say that the function M satisfies the ∇_2-condition, if*

$$\lim_{u\to\infty}\inf\frac{M(2u)}{M(u)}>2,\ i.e.\ \ \exists\lambda>2\ \text{and}\ \exists u_0>0:\ M(2u)\geq\lambda M(u),\quad\forall u\geq u_0.$$

Denote by $\Delta_2(\infty)$ ($\nabla_2(\infty)$) the set of all N-functions satisfying the Δ_2-condition (the ∇_2-condition).

We will need the concepts of Boyd indices of Orlicz spaces. M^{-1} denotes the inverse of the N-function M. Assume

$$h(t)=\lim_{x\to\infty}\sup\frac{M^{-1}(x)}{M^{-1}(tx)},\quad t>0.$$

Define the following numbers

$$\alpha_M=-\lim_{t\to\infty}\frac{\log h(t)}{\log t}\ ;\qquad \beta_M=-\lim_{t\to0^+}\frac{\log h(t)}{\log t}.$$

The numbers α_M and β_M are known as lower and upper Boyd indices for the Orlicz space L_M, respectively. These numbers satisfy the following relations

$$0\leq\alpha_M\leq\beta_M\leq1\ \ ;\qquad \alpha_M+\beta_{M^*}=1;\qquad \alpha_{M^*}+\beta_M=1,$$

where M and M^* are complementary N-functions each to other.

Theorem 1. *The Orlicz space L_M is reflexive if and only if the relation $0<\alpha_M\leq\beta_M<1$ holds. Moreover, if for the numbers $p,\ q\in[1,+\infty]$, the inequalities*

$$1\leq q<\frac{1}{\beta_M}\leq\frac{1}{\alpha_M}<p\leq+\infty,$$

hold, then the continuous embeddings

$$L_p\ \hookrightarrow\ L_M\ \hookrightarrow\ L_q,$$

are valid.

More information about these and other facts can be found in works [22, 23, 24].

Let's give the conjugate function $\widetilde{f}$ of function f from the Orlicz Space $L_M(I)$.

Definition 9. *For any $f\in L_M(I)\subset L_1(I)$, the conjugate function $\widetilde{f}$ of f is defined by*

$$\widetilde{f}(x)=-\frac{1}{\pi}\int_0^{\pi}\frac{f(x+t)-f(x-t)}{2\tan\frac{t}{2}}\,dt.$$

We denote the partial sum of Fourier series of function $f \in L_M(I)$ by

$$S_n[f](x) = \sum_{|k| \leq n} c_k \, e^{ikx} = \frac{1}{\pi} \int_0^{2\pi} f(t) \, D_n(x-t) \, dt, \quad n = 0, 1, ...,$$

where

$$c_k = c_k(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, e^{-ikx} \, dx, \quad k \in \mathbb{Z},$$

are Fourier coefficients of $f(.)$ and

$$D_n(x) = \frac{1}{2} \sum_{|k| \leq n} e^{ikx} = \frac{\sin\left[\left(x + \frac{1}{2}\right) x\right]}{2 \sin \frac{x}{2}}, \quad n = 0, 1, ...,$$

is a Dirichlet kernel of order n.

We need the following Ryan's theorem from the monographs [23, 25].

Theorem 2 (Ryan)**.** *Let M be an N-function. Then the following statements are equivalent:*

(i) $L_M(I)$ is reflexive;

(ii) There is a constant $C > 0$ such that for all $f \in L_M(I)$:

$$\|\widetilde{f}\|_{L_M(I)} \leq C \, \|f\|_{L_M(I)};$$

(iii) There is a constant $C > 0$ such that for all $n \geq 1$ and $f \in L_M(I)$:

$$\|S_n[f]\|_{L_M(I)} \leq C \, \|f\|_{L_M(I)}.$$

The following conclusions directly follow from these facts.

Corollary 1. *For N-function M:*

$$\lim_{n \to \infty} \|S_n[f] - f\|_{L_M(I)} = 0,$$

for all $f \in L_M(I)$ if and only if $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$.

The following Ryan's theorem is also valid.

Theorem 3 (Ryan)**.** *Let M be an N-function. If holds the part (iii) of Theorem 2 (Ryan), then $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$; so $L_M(I)$ is reflexive [23].*

Taking into account the Theorems 2, 3 and Corollary 1, we arrive to the following conclusion.

Corollary 2. *Let M be an N-function. Then the Boyd indices of Orlicz space $L_M(I)$: α_M; $\beta_M \in (0,1)$ if and only if $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$.*

We define the Sobolev space generated by the Orlicz space on unbounded rectangle $\Pi = (0,\, 2\pi) \times (0, \infty)$. Let M be some N-function. First define the space $L_M(\Pi)$ by the norm

$$\|u\|_{L_M(\Pi)} = \int_0^\infty \|u(\cdot;y)\|_{L_M(I)}\, dy.$$

The corresponding Sobolev space is defined by the relation

$$W_M^2(\Pi) = \{u \in L_M(\Pi) : \ \partial^\alpha u \in L_M(\Pi),\ \forall \alpha;\ |\alpha| \le 2\},$$

and provide it with the norm

$$\|u\|_{W_M^2} = \sum_{|\alpha| \le 2} \|\partial^\alpha u\|_{L_M(\Pi)}.$$

We will also consider the Sobolev space $W_M^2(I)$ generated by the norm

$$\|f\|_{W_M^2(I)} = \|f\|_{L_M(I)} + \left\|f'\right\|_{L_M(I)} + \left\|f''\right\|_{L_M(I)}.$$

2.4. Trace Operator

For correct statement of considered boundary value problem in Orlicz-Sobolev space we need firstly define the trace operator corresponding to the space $W_M^1(\Omega)$ on the bounded domain $\Omega \subset \mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega$. Let $\alpha_M, \beta_M \in (0,1)$. Then it is evident that it is true the continuous embedding $W_M^1(\Omega) \hookrightarrow W_1^1(\Omega)$, where $W_p^k(\Omega)$ usually denotes classical Sobolev space. Let $L \subset \overline{\Omega}$ be some smooth line and dl is the length element of L. Denote by $T_L \in \left[W_1^1(\Omega); L_1(L;dl)\right]$ trace operator, corresponding to L (existence this operator very known and regarding this question one can see f. e. [2, 26]). Therefore, if $|L| < +\infty$ ($|L|$ is length of L), then for $\forall u \in W_M^1(\Omega)$ we have

$$\|T_L u\|_{L_1(L;dl)} \le \ c\, \|u\|_{W_1^1(\Omega)} \le \ c\, \|u\|_{W_M^1(\Omega)}.$$

Consequently, $T_L \in \left[W_M^1(\Omega); L_1(L;dl)\right]$. Based on this fact T_L we will call as the trace operator regarding the Orlicz-Sobolev space $W_M^1(\Omega)$ corresponding to L. According to this concept accept

$$\Gamma_0 = \{(0;y) : \ y \in (0,\infty)\}, \quad \Gamma_{2\pi} = \{(2\pi;y) : \ y \in (0,\infty)\}.$$

So, $\partial\Pi = I \cup \Gamma_0 \cup \Gamma_{2\pi}$. The trace operators corresponding to subsets I, Γ_0 and $\Gamma_{2\pi}$ we denote by T_I, T_0 and $T_{2\pi}$, respectively. Let us define these operators. Set

$$\Gamma_0^{(n)} = \{(0;y) : \; y \in (0,n)\}, \; \Gamma_{2\pi}^{(n)} = \{(2\pi;y) : \; y \in (0,n)\}, \; \Pi^{(n)} = I \times (0,n), \;\; n \in \mathbb{N}.$$

Let

$$u \in W_M^1(\Pi) \Rightarrow u \in W_M^1(\Pi^{(n)}), \; \forall n \in \mathbb{N}.$$

Consequently

$$\varphi_n = T_{\Gamma_0^{(n)}} u \in L_1\left(\Gamma_0^{(n)}; dy\right), \quad \psi_n = T_{\Gamma_{2\pi}^{(n)}} u \in L_1\left(\Gamma_{2\pi}^{(n)}; dy\right), \quad \forall n \in \mathbb{N}.$$

It is evident that

$$\varphi_{n+1}|_{\Gamma_0^{(n)}} = \varphi_n, \quad \psi_{n+1}|_{\Gamma_{2\pi}^{(n)}} = \psi_n, \quad \forall n \in \mathbb{N}.$$

Based on these relations define the function $\varphi(\cdot)$ on Γ_0 ($\psi(\cdot)$ on $\Gamma_{2\pi}$) by expression $\varphi(\xi) = \varphi_n(\xi)$, if $\xi \in (0,n)$ ($\psi(\xi) = \psi_n(\xi)$, if $\xi \in (0,n)$). It is obvious that

$$\varphi \in L_1^{loc}(0,+\infty), \quad \psi \in L_1^{loc}(0,+\infty).$$

Thus, the trace operators $T_0 : W_M^1(\Pi) \to L_1^{loc}(\Gamma_0)$ and $T_{2\pi} : W_M^1(\Pi) \to L_1^{loc}(\Gamma_{2\pi})$ define as $T_0 u = \varphi$, $T_{2\pi} u = \psi$. It is evident that

$$T_0 \in \left[W_M^1(\Pi); L_1(\Gamma_0^{(n)})\right], \quad T_{2\pi} \in \left[W_M^1(\Pi); L_1(\Gamma_{2\pi}^{(n)})\right], \quad \forall n \in \mathbb{N}.$$

Now we consider the following Problem A

$$\Delta u\,(x;y) = 0\,, \quad (x;y) \in \Pi, \tag{5}$$

$$T_I u = f\,, \quad T_0 u = T_{2\pi} u\,, \quad T_0(\partial_x u) = 0, \tag{6}$$

where $f \in L_M(I)$ is a given function.

Definition 10. *A function $u \in W_M^2(\Pi)$ is called a strong solution of the Problem A, if the equality* (5) *is satisfied for a.e.* $(x;y) \in \Pi$ *and regarding its trace* $u|_{\partial\Pi}$ *it is true the relations* (6).

2.5. Basicity of root functions

Applying the Fourier method to solution of the Problem A leads to the following spectral problem

$$\left.\begin{array}{c} y''(x) + \lambda^2 y(x) = 0, \quad x \in I, \\ y(0) = y(2\pi); \quad y'(0) = 0. \end{array}\right\}$$

The eigenvalues of this problem are $\lambda_n = n,\ n \in \mathbb{Z}_+$, and corresponding eigenfunctions are $\{y_n^c(x) = \cos nx\}_{n\in\mathbb{Z}_+}$. Each eigenfunction $y_n,\ n \in \mathbb{N}$, has one associated function $y_n^s(x) = x\sin nx,\ n \in \mathbb{N}$. Consider the collection of root functions

$$y_0^c = 1;\quad y_n^c(x) = \cos nx;\quad y_n^s(x) = x\sin nx,\ n \in \mathbb{N}, \tag{7}$$

and also set

$$\vartheta_0^c(x) = \frac{2\pi - x}{2\pi^2};\quad \vartheta_n^c(x) = \frac{2\pi - x}{\pi^2}\cos nx;\quad \vartheta_n^s(x) = \frac{1}{\pi}\sin nx,\ n \in \mathbb{N}. \tag{8}$$

Let us prove the following theorem.

Theorem 4. *Let M be an N-function and the Boyd indices of Orlicz space $L_M(I):\ \alpha_M,\ \beta_M \in (0,1)$. Then the system (7) forms a basis in $L_M(I)$.*

Proof. Consider the following functionals

$$e_0^c(f) = \frac{1}{2\pi^2}\int_0^{2\pi} f(x)(2\pi - x)\,dx;\quad e_n^c(f) = \frac{1}{\pi^2}\int_0^{2\pi} f(x)(2\pi - x)\cos nx\,dx;$$

$$e_n^s(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx,\qquad n \in \mathbb{N}.$$

In the work [8] the following relations

$$\begin{array}{llll} e_n^c(y_m^c) = \delta_{nm}; & \forall n,m \in \mathbb{N}; & e_n^c(y_m^s) = 0, & \forall n \in \mathbb{Z}_+;\ \forall m \in \mathbb{N}; \\ e_n^s(y_m^c) = 0, & \forall n \in \mathbb{N};\ \forall m \in \mathbb{Z}_+; & e_n^s(y_m^s) = \delta_{nm}; & \forall n,m \in \mathbb{Z}_+, \end{array}$$

are established. Let us show that the functionals $\{e_n^c;\ e_n^s\}$ belong to the space $(L_M(I))^*$. It is evident that $\exists p,q \in (1,+\infty)$, for which the inequalities

$$1 < q < \frac{1}{\beta_M} \le \frac{1}{\alpha_M} < p < +\infty,$$

valid. Then from the embeddings in Theorem 1, the following estimates

$$\|f\|_{L_q(I)} \le c\,\|f\|_{L_M(I)};\quad \forall f \in L_M(I),$$

follow, where $c > 0$ some constant.

Applying the Hölder inequality we have

$$|e_n^c(f)| \le c \int_0^{2\pi} |f(x)(2\pi - x)\cos nx|\, dx \le c \left(\int_0^{2\pi} |f|^q dx \right)^{\frac{1}{q}} \le c\|f\|_{L_M(I)}, \quad \forall n \in \mathbb{Z}_+,$$

and also

$$|e_n^s(f)| \le c \int_0^{2\pi} |f| dx \le c \left(\int_0^{2\pi} |f|^q dx \right)^{\frac{1}{q}} \le c\|f\|_{L_M(I)}, \quad \forall n \in \mathbb{N},$$

where c denotes constants. From this, it immediately follows that

$$\{e_n^c;\ e_n^s\} \subset (L_M(I))^*.$$

Then based on minimality criterion from relations (3) we have the minimality of the system (7) in $L_M(I)$.

Let us prove the completeness of the system (7) in $L_M(I)$. From Corollary 2 it follows that $M \in \Delta_2(\infty) \cap \nabla_2(\infty)$ and as a result, it is known that (see, e.g. the monographs [22, 23]) the class $C_0^\infty(I)$ is dense in $L_M(I)$. Let $f \in L_M(I)$ be an arbitrary function. Take $\forall \varepsilon > 0$. Then $\exists\, g \in C_0^\infty(I)$, such that $\|f - g\|_{L_M(I)} < \varepsilon$. Let us consider the biorthogonal series of g on the system (7):

$$\widetilde{S}_n[g](x) = \sum_{k=0}^{n} e_k^c(g)\ y_k^c(x) + \sum_{k=0}^{n} e_k^s(g)\ y_k^s(x), \quad n \in \mathbb{N}.$$

Consider the biorthogonal coefficients $\{e_n^c;\ e_n^s\}$:

$$e_k^c(g) = c \int_0^{2\pi} g(x)(2\pi - x)\cos kx dx = \int_0^{2\pi} \widetilde{g}(x) \cos kx dx, \quad k \in \mathbb{Z}_+,$$

where $\widetilde{g}(x) = c\ g(x)(2\pi - x)$ and c denotes the corresponding coefficient of the biorthogonal system. It is evident that $\widetilde{g} \in C_0^\infty(I)$ and as a result $\widetilde{g}^{(n)}(0) = \widetilde{g}^{(n)}(2\pi) = 0,\ \forall n \in \mathbb{Z}_+$. Integrating by parts twice and taking into account these relations, we have

$$e_k^c(g) = \frac{1}{k} \int_0^{2\pi} \widetilde{g}(x) d \sin kx = -\frac{1}{k} \int_0^{2\pi} \widetilde{g}^{(1)}(x) \sin kx dx = -\frac{1}{k^2} \int_0^{2\pi} \widetilde{g}^{(2)}(x) \cos kx\ dx,$$

and from this, it follows

$$|e_k^c(g)| \le \frac{c}{k^2}, \quad \forall k \in \mathbb{N}.$$

Completely analogously we have the following estimate

$$|e_k^s(g)| \le \frac{c}{k^2}, \quad \forall k \in \mathbb{N}.$$

From these estimates, it follows that the partial sums $\{\widetilde{S}_n[g]\}_{n\in\mathbb{N}}$ converges uniformly on $\overline{I}$. By results of the work [8] the system (7) forms a basis in $L_2(I)$ and as a result it is evident that the limit of sums $\{\widetilde{S}_n[g]\}_{n\in\mathbb{N}}$ is $g(.)$. It is obvious that $\exists c > 0$:

$$\|f\|_{L_M(I)} \ \le \ c \ \|f\|_{L_\infty(I)}; \ \ \forall f \in C(\overline{I}).$$

Then $\exists n_\varepsilon \in \mathbb{N}$ such that for $\forall n \ge n_\varepsilon$, it holds

$$\|\widetilde{S}_n[g] - g\|_{L_M(I)} \ \le \ c \ \|\widetilde{S}_n[g] - g\|_{L_\infty(I)} \ < \ \varepsilon.$$

We have

$$\|f - \widetilde{S}_n[g]\|_{L_M(I)} \ \le \ \|f - g\|_{L_M(I)} \ + \ \|\widetilde{S}_n[g] - g\|_{L_M(I)} \ < \ 2\varepsilon, \ \ \forall n \ge n_\varepsilon.$$

From the arbitrariness of $\varepsilon > 0$ follows the completeness of the system (7) in $L_M(I)$.

For establishing basicity of the system (7) in $L_M(I)$ it is sufficient to prove that the projectors

$$P_n(f) \ = \ \sum_{k=0}^{n} e_k^c(f) \ y_k^c \ + \sum_{k=1}^{n} e_k^s(f) \ y_k^s, \quad \forall n \in \mathbb{N},$$

are uniformly bounded in $L_M(I)$. We have

$$\|P_n(f)\|_{L_M(I)} \ \le \ \left\| \sum_{k=0}^{n} e_k^c(f) \ y_k^c \right\|_{L_M(I)} + \left\| \sum_{k=1}^{n} e_k^s(f) \ y_k^s \right\|_{L_M(I)} = I_n^{(1)} + I_n^{(2)}, \quad n \in \mathbb{N}.$$

Let us estimate $\{I_n^{(1)}\}$. We have

$$e_k^c(f) = c_k^+(\widetilde{f}), \quad \forall k \in \mathbb{Z}_+,$$

where $c_k^+(\widetilde{f})$ is the Fourier coefficient

$$c_k^+(\widetilde{f}) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \widetilde{f}(x) \ \cos kx \ dx,$$

of the function $\widetilde{f}(x) = c\ (2\pi - x) f(x)$. Since the classical trigonometric system $\{1; \cos nx; \sin nx\}_{n\in\mathbb{N}}$ forms a basis in $L_M(I)$ (it follows from Corollary 1), then from the basicity criterion, it follows that

$$I_n^{(1)} \ = \ \left\| \sum_{k=0}^{n} c_k^{+}(\widetilde{f}) \ \cos kx \right\|_{L_M(I)} \ \leq \ c\ \|\widetilde{f}\|_{L_M(I)} \ \leq \ c\ \|f\|_{L_M(I)},$$

where the constant $c > 0$ does depend on n and f. Completely analogously we can establish

$$I_n^{(2)} \ \leq \ c\ \|f\|_{L_M(I)}, \quad \forall n \in \mathbb{N}.$$

As a result from the basicity criterion, it follows that the system (7) forms a basis in $L_M(I)$. The theorem is proved.

3. Main Results

In this section, we will examine the uniqueness and existence of the strong solution of Problem A in Orlicz-Sobolev space. Here we will take the strong solution in the sense of Definition 10.

Let us first prove the uniqueness of the solution.

Theorem 5. *Let M be N-function and the Boyd index of L_M be $\beta_M \in (0,1)$ and $f \in L_M(I)$. Then if the Problem A has a solution in $W_M^2(\Pi)$, then it is unique.*

Proof. Indeed, in this case it is evident that $\exists p \in (1, +\infty) :\ L_M(I) \hookrightarrow L_p(I)$ and as a result by definition of $W_M^2(\Pi)$ we have $W_M^2(\Pi) \hookrightarrow W_p^2(\Pi)$, where the Sobolev space $W_p^2(\Pi)$ is generated by the norm

$$\|u\|_{L_p(\Pi)} = \int_0^{+\infty} \|u(\cdot; y)\|_{L_p(I)}\ dy,$$

where

$$\|f\|_{L_p(I)} = \left(\int_I |f|^p\ dt \right)^{\frac{1}{p}}.$$

Then from results of the works [16, 27] we obtain the uniqeness of the solution in $W_M^2(\Pi)$. The theorem is proved.

Theorem 6. *Let M be an N-function and the Boyd indices of L_M: $\alpha_M, \beta_M \in (0,1)$ and the boundary function f satisfies*

$$f \in W_M^2(I), \quad f(2\pi) - f(0) = f'(0) = 0.$$

Then the Problem A has a (unique) solution in $W_M^2(\Pi)$.

Proof. Let $p \in (1, +\infty)$ such that $L_M(I) \hookrightarrow L_p(I)$, and therefore $W_M^2(\Pi) \hookrightarrow W_p^2(\Pi)$. Without loss of generality assume that $p \in (1,2)$. Let $u \in W_M^2(\Pi)$ be the solution of the Problem A. It is evident that u is also a solution of the Problem A in the sense of $W_p^2(\Pi)$, since $f \in L_p(I)$. Then, by the results of the works [16, 27], the solution $u \in W_p^2(\Pi)$ has the following representation

$$u\,(x;y) = u_0\,(y) + \sum_{n=1}^{\infty}(u_n\,(y)\,\cos nx + v_n\,(y)\,x \sin nx)\ ,\ \ (x,y) \in \Pi,$$

where the coefficients $u_0\,(\cdot)\,,\ u_n\,(\cdot)\,,\ v_n\,(\cdot)\,,\, n \in \mathbb{N}$, are defined by the relations

$$\left.\begin{aligned} u_0\,(y) &= \tfrac{1}{2\pi^2}\int_0^{2\pi} u\,(x,y)\ (2\pi - x)\ dx, \\ u_n\,(y) &= \tfrac{1}{\pi^2}\int_0^{2\pi} u\,(x,y)\ (2\pi - x)\ \cos nx\, dx, \\ v_n\,(y) &= \tfrac{1}{\pi}\int_0^{2\pi} u\,(x,y)\ \sin nx\, dx,\ \ n \in N. \end{aligned}\right\}$$

Let us show that the function $u\,(x,y)$ belongs to $W_M^2\,(\Pi)$, first consider the series

$$u_1\,(x,y) = \sum_{n=1}^{\infty} v_n\,(y)\ x\ \sin nx.$$

Differentiating this series formally term-by-term, we have

$$\frac{\partial^2 u_1}{\partial y^2} = \sum_{n=1}^{\infty} v_n''\,(y)\ x\ \sin nx = \sum_{n=1}^{\infty} n^2 v_n\,(y)\ x\ \sin nx, \tag{9}$$

$$\frac{\partial u_1}{\partial x} = \sum_{n=1}^{\infty} v_n\,(y)\ \sin nx + \sum_{n=1}^{\infty} n v_n\,(y)\ x\ \cos nx, \tag{10}$$

$$\frac{\partial^2 u_1}{\partial x^2} = 2\sum_{n=1}^{\infty} n v_n\,(y)\ \cos nx - \sum_{n=1}^{\infty} n^2 v_n\,(y)\ x\ \sin nx. \tag{11}$$

Denote

$$w\,(x,y)=\sum_{n=1}^{\infty}n^{2}v_{n}\,(y)\;x\;\sin nx.$$

Let us show that the function $w\,(x,y)$ belongs to $L_M\,(\Pi)$. It is evident that the Newton-Leibniz formula

$$u(x;y+h)-u(x;y)=\int\limits_{y}^{y+h}\frac{\partial u(x;t)}{\partial t}dt,\quad \forall y>0,$$

holds for a.e. $x\in I$. As already established, from $\alpha_M,\beta_M\in(0,1)$ it follows that $L_M(\Pi)\hookrightarrow L_1(\Pi)$. Therefore, $\frac{\partial u}{\partial y}\in L_1(\Pi)$, and as a result, by Theorem 1.1.1 from [28, p. 13], it follows that the functions $\{u_n,v_n\}$ are twice differentiable and can be differentiated under the integral sign. Multiplying the equation by $\sin nx$ and integrating it over I, for $v_n(\cdot)$ we obtain the relation

$$v_n''(y)-n^2v_n(y)=0,\quad y>0. \tag{12}$$

Let $\alpha\in C^\infty(\mathbb{R})$ be such that $\alpha(y)\equiv 1$ in a sufficiently small neighborhood of the point $y=0$ and $\alpha(y)=0$, $\forall y:|y|\geq 1$. Considering the function $F(x;y)=\alpha(y)u(x;y)$, we get $F(x;y)=0$, $\forall y\geq 1$. Therefore, without loss of generality, we will assume $u(x;y)=0$, $\forall y\geq 1$ in the calculations below. So, we have

$$u(x;y)=-\int\limits_{y}^{1}\frac{\partial u(x;t)}{\partial t}dt,\ \text{ a.e. } x\in I,\quad \Rightarrow f(x)=u(x;0)=-\int\limits_{0}^{1}\frac{\partial u(x;t)}{\partial t}dt,\ \text{ a.e. } x\in I.$$

Consequently

$$|u(x;y)-f(x)|\leq\int\limits_{0}^{y}\left|\frac{\partial u(x;t)}{\partial t}\right|dt,\quad \text{a.e. } x\in I,$$

and from this it immediately follows

$$\int\limits_{I}|u(x;y)-f(x)|dx\leq\int\limits_{I}\int\limits_{0}^{y}\left|\frac{\partial u(x;t)}{\partial t}\right|dtdx.$$

Since $|\{(x;y):(x;y)\in I\times(0,n)\}|\to 0,\ n\to 0^+$, it is clear that $u(\cdot;y)\to f(\cdot),\ y\to 0^+$, in $L_1(I)$. It is easy to see that $v_n(\cdot)\in W_1^2(0,+\infty)$, and it follows that $\exists\lim_{y\to 0^+}v_n(y)=v_n(0)$, $\forall n\in\mathbb{N}$. From these relations we directly obtain

$$v_n(0) = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx, \quad \forall n \in \mathbb{N}. \tag{13}$$

Also we have

$$v_n(y) - v_n(0) = \frac{1}{\pi} \int_0^{2\pi} (u(x;y) - u(x;0)) \sin nx dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^{y} \frac{\partial u(x;t)}{\partial t} \sin nx dt dx \implies |v_n(y) - v_n(0)| \leq \frac{1}{\pi} \iint_{\Pi} \left| \frac{\partial u}{\partial y} \right| dx dy < +\infty.$$

From here we directly obtain

$$\sup_{y>0} |v_n(y)| < +\infty. \tag{14}$$

The solution to problem (12)-(14) is

$$v_n(y) = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \ e^{-ny}, \quad \forall n \in \mathbb{N}.$$

Similarly for u_n, we obtain

$$u_0(y) = \frac{1}{2\pi} \int_0^{2\pi} (2\pi - x) f(x) dx,$$

$$u_n(y) = \frac{1}{\pi^2} \int_0^{2\pi} (2\pi - x) f(x) \cos nx dx e^{-ny} + \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \, y e^{-ny}, \quad \forall n \in \mathbb{N}.$$

Suppose

$$f_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \ dx.$$

Consequently $v_n(y) = f_n e^{-ny}, \quad n \in \mathbb{N}$. If we apply integration by parts, we get

$$f_n = -\frac{1}{n\pi} \int_0^{2\pi} f(x) \cos nx \ dx = -\frac{1}{n\pi} \left(f(2\pi) - f(0) - \int_0^{2\pi} f'(x) \cos nx \ dx \right)$$

$$= \frac{1}{n\pi} \int_0^{2\pi} f'(x) \cos nx \; dx = \frac{1}{n^2\pi} \int_0^{2\pi} f''(x) \sin nx \; dx = \frac{1}{n^2} f''_n,$$

where

$$f''_n = (f'')_n = \frac{1}{\pi} \int_0^{2\pi} f''(x) \sin nx \; dx.$$

Thus

$$w(x;y) = \sum_{n=1}^{\infty} f''_n x \sin nx \; e^{-ny}.$$

So, using the embeddings

$$L_{p'}(I) \hookrightarrow L_M(I) \hookrightarrow L_p(I),$$

and applying the classical Hausdorff-Young theorem (see, e. g. [29]) we obtain

$$\|w(\cdot;y)\|_{L_M} \le c \left(\int_0^{2\pi} |w(x;y)|^{p'} \; dx \right)^{\frac{1}{p'}} \le c \left(\sum_{n=1}^{\infty} \left| f''_n e^{-ny} \right|^p \right)^{\frac{1}{p}} \le c \sum_{n=1}^{\infty} \left| f''_n e^{-ny} \right|.$$

In the last part we have applied the inequality $\left(\sum_{n=1}^{\infty} |a_n|\right)^{\alpha} \le \sum_{n=1}^{\infty} |a_n|^{\alpha}$, which is true for every $0 < \alpha \le 1$.

This leads to the following estimate

$$\|w\|_{L_M(\Pi)} \;\le\; c \sum_{n=1}^{\infty} \left| f''_n \right| \int_0^{\infty} e^{-ny} \; dy \;=\; c \sum_{n=1}^{\infty} \frac{|f''_n|}{n}.$$

Applying again Hölder inequality we get

$$\|w\|_{L_M(\Pi)} \;\le\; c \; \left(\sum_{n=1}^{\infty} \frac{1}{n^p} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |f''_n|^{p'} \right)^{\frac{1}{p'}} \;=\; c \; \left(\sum_{n=1}^{\infty} |f''_n|^{p'} \right)^{\frac{1}{p'}}.$$

It is evident that $p' \ge 2$. Then applying the classical Hausdorff-Young inequality again, we obtain

$$\|w\|_{L_M(\Pi)} \;\le\; c \left(\sum_{n=1}^{\infty} \left| f''_n \right|^{p'} \right)^{\frac{1}{p'}} \;\le\; c \; \|f''\|_{L_p(I)} \;\le\; c \; \|f''\|_{L_M(I)}.$$

Other series from (9)-(11), and therefore all series from the expression $u(\cdot,\cdot)$ are evaluated in a similar manner. Hence it follows that

$$\|u\|_{W^2_M(\Pi)} \leq c\, \|f''\|_{L_M(I)},$$

where $c > 0$ is a constant independent of f.

Let us verify that the boundary conditions are satisfied. We show that $T_I u = f$. From the boundedness of the operator $T_I \in \left[W^2_M(\Pi)\ ;\ L_1(I)\right]$, it follows that if $u_m \to u$ in $W^2_M(\Pi)$, then $u_m|_I \to u|_I$ in $L_1(I)$.

Now, consider the following functions

$$u_m(x,y) = u_0(y) + \sum_{n=1}^{m} \left(u_n(y)\cos nx + v_n(y)\, x\sin nx\right),\ \ (x;y) \in \Pi, m \in N.$$

We have

$$T_I u_m = u_m(x,0) = u_0(0) + \sum_{n=1}^{m} \left(u_n(0)\cos nx + v_n(0)\, x\sin nx\right) = \frac{1}{2\pi^2}\int_0^{2\pi} f(x)(2\pi - x)\,dx$$

$$+\sum_{n=1}^{m}\left(\frac{1}{\pi^2}\int_0^{2\pi} f(x)(2\pi - x)\cos nx dx\ \cos nx + \frac{1}{\pi^2}\int_0^{2\pi} f(x)\sin nx dx\ x\sin nx\right).$$

The basicity of the system (7) for $L_M(I)$ implies $T_I u_m \to f\ ,\ m \to \infty$, in $L_M(I)$. Consequently, $T_I u = f$.

Consider the operators T_0 and $T_{2\pi}$. Let $J_n = (0,n)$, $n \in \mathbb{N}$.. It is not hard to see that $(T_0 u_m)|_{J_n} = (T_{2\pi} u_m)|_{J_n}$, $\forall m; n \in \mathbb{N}$.. Since $T_0; T_{2\pi} \in \left[W^1_M(\Pi); L_1(J_n)\right]$ and $u_m \to u$, $m \to \infty$ in $W^1_M(\Pi)$, then it is evident that $(T_0 u)|_{J_n} = (T_{2\pi} u)|_{J_n}$, $\forall n \in \mathbb{N}$.. Therefore, $T_0 u = T_{2\pi} u$. Completely analogously, we establish that $T_0(\partial_x u) = 0$. The theorem is proved.

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