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Analysis of Queuing-Inventory System MAP/PH/1 with Instantaneous Feedback

Agassi Melikov*, Dhanya Shajin, Sevinj Aliyeva, Janos Sztrik

Abstract. This paper considers single server queuing-inventory models with instantaneous feedback scheme. The primary customers (p-customers) arrive according to a Markovian arrival process and service times have PH distribution. Both models with an infinite and finite common buffer for waiting of the arrived p-customers and feedback customers (f-customers) are investigated. It is assumed that serving each type of customer requires the same amount of inventory. If the inventory level is zero at the arrival moment of p-customer, then according to Bernoulli trial, the p-customer either joins the queue or leaves the system. After the service completion, the customer may decide either to join the common queue (feedback) or leave the system according to a Bernoulli trial where parameters of Bernoulli trial are depending on current inventory level. Multiple feedback is allowed. The customers in the queue become impatient when the inventory level is zero and they independently leave the system after waiting some exponentially distributed period of time. An (s, S)-type inventory replenishment policy is used and the lead time is assumed to have exponential distribution. The key performance measures of the system are determined using the matrix-geometric method, and the long-term total expected revenue rate is calculated and maximized. The results are illustrated numerically.

Key Words and Phrases: Queueing, Inventory, Feedback, Impatience, Matrix-geometric method

2010 Mathematics Subject Classifications: 60J28, 60K25, 90B05, 90B22

1. Introduction and Motivation

Recent review papers by Krishnamoorthy et al. [19] and Salini et al. [29] note that the first studies on queuing-inventory systems (QIS) were published independently by Sigman and Simchi-Levi [31] and Melikov and Molchanov [23].

*Corresponding author.

The problems of QIS theory have attracted much attention from researchers over the past three decades, and the relevant literature can be found in the detailed reviews mentioned above.

Besides, we refer the reader to earlier review papers by Krishnamoorthy et al. [16], Krishnamoorthy et al. [17], and Bijvanik and Vis [6], as well as to the research papers of Chakravarthy and Hayat [9], Keerthana et al. [15], Jeganathan et al. [12], Sugapriya et al. [33], Mirzayev [24] and Poladova and Valijanova [27] and the references therein.

QIS may be considered from two viewpoints: from viewpoint of queuing system (QS) and from viewpoint of inventory control system (ICS). In fact, QIS may be considered as QS in which for serving of customer, besides empty server, positive level of inventory is required and at the end of service process, the inventory level is decreased. On the other hand, QIS can be viewed as an ICS with attached server, in which the service time of customers is a positive random variable. In other words, QIS combines the properties of both classical queuing and inventory control systems. Such kind of systems have many applications in real life situation, i.e. in manufacturing, supply chain management, operations management, energy harvesting in the wireless network, telecommunications, etc.

Since QIS are a combination of QS and ICS, it is necessary to study QIS models in which different features of QS and ICS occur within a single system. Based on the objective of this paper, we consider here QIS models that take into account the feedback effect from already served consumer customers. In the available literature, QIS models with feedback have been little studied. Note that, first papers devoted to classical QS with feedback concept were published by Takacs ([34], [35]). In recent years, many authors have extensively investigated models of classical QSs with the concept of feedback under various assumptions (e.g. with repeated attempts of customers, with server vacations, with homogeneous and heterogeneous servers, with impatient customers, with batch arrivals, etc.). Here we will not consider in detail the known results related to models of classical QS with feedback phenomena, but will refer to some recent works, see Ayyappan and Karpagam [4], Ayyappan and Thilagavathy [5], Bouchentouf et al. [7], Bouchentouf and Guendouzi [8], Chakravarthy et al. [10], Jain and Kaur [11], Ke et al. [14], Krishnamoorthy and Manjunath [18], Melikov et al. ([20], [21]), Rajadurai et al. [28], Varshney et al. [36] and Zhao and Yue [37]), in which further literature can be found.

Although the feedback effect in QIS is quite natural, models of such systems have not been sufficiently studied. Indeed, for example, having bought a certain amount of gasoline, some consumers living near a specific gas station return again after some time to buy a new batch of gasoline. A similar situation occurs in all shopping centers. This type of feedback can be considered as “positive” feedback where a customer returns to the system due to previous good service. Sometimes we see "negative" feedback when a customer returns to the system due to previous poor service. In the latter case, the customer either requires repeated service or returns the purchased items, see Shajin and Melikov [30] and Jeganathan et al. [13] and their reference lists. To the best of our knowledge, there are only a few studies that consider queuing-inventory systems with feedback (QISwFB). Let us briefly review these works.

For brevity, here we consider the following symbolic notation for QISwFB. We propose to add symbols to the classical Kendall notation for QS systems to indicate the replenishment policy applied in the system, the type of lead time distribution function, and the type of feedback scheme, i.e., instantaneous feedback (IFB) and/or delayed feedback (DFB).

First, let's consider QIS models with "positive" feedback, i.e. without returns of purchased items. It seems that first contributions to QISwFB models were the works done by Amirthakodi and Sivakumar [2] and Amirthakodi et al. [1]. The model of retrial M/M/1/N/(s, S)/M/DFB was investigated by Amirthakodi and Sivakumar [2]. It means that, after a customer is served, it will decide either to join the finite size retrial group (called orbit) for another service or leave the system according to a Bernoulli trial. Sojourn times of feedback customers (f-customers) in orbit, as well as their service times are assumed to be independent and exponential distribution with finite parameters. F-customers compete for service according to linear retrial rate. It is assumed that f-customer joins the server if there are no primary customers (p-customers) in the system or the inventory level is zero or both. The f-customer does not demand item, i.e. they only require the service. Non-preemptive priority rule is used in the system, i.e. the p-customer arrived during the f-customer service will wait up to the service completion of f-customer. Multiple feedback is allowed, i.e. after the service completion to the f-customer, it will decide whether to join the orbit for additional service or leave the system according to a Bernoulli trial. The algorithm to calculate the joint probability distribution of the inventory level, the number of f-customers in the orbit, the number of p-customers in the waiting room and the status of the server is developed. Key system performance measures in the steady state are derived, and the long-run total expected cost rate is also calculated. In addition, the Laplace-Stieljes transforms of waiting time distribution of both primary and feedback customers are obtained, and the sensitivity analysis is performed to examine the effect of different parameters and cost in the system. Same model with infinite capacity of the orbit and constant retrial rate of f-customers is considered in Amirthkodi and Sivakumar [3].

The model of perishable MAP/M/1/N/(s,S)/PH/IFB was investigated in the paper by Amirthakodi et al. [1]. The life time of each item has an exponential distribution with finite parameter. At the time of service completion, the p-customer may in accordance of Bernoulli scheme decide either to join the secondary (feedback) queue, which is of infinite size for additional service or leave the system. After the service completion of p-customer, the server in accordance of Bernoulli will decide either to serve the p-customer or the f-customer. The primary and feedback customer's service counters are different. Unlike the Amirthakodi and Sivakumar ([2], [3]), here, multiple IFB scheme is used. After completing a service of f-customer, the server immediately starts p-customer's service, provided the inventory level is positive and there are p-customers in the system. On return from the secondary counter, if the server finds an empty stock or no p-customers are in the waiting hall, it goes to vacation that has an exponential distribution. During the vacation period, if the number of p-customers and inventory level become positive, the server starts service of p-customers immediately. At the end of the vacation, if either the inventory level or the number of p-customers is zero, then it goes to secondary queue to serve an f-customer, if there is any. As in Amirthakodi and Sivakumar ([2], [3]), the f-customers request only for service, but not an item. The steady state analysis of the model is done by using matrix-geometric method (MGM) by Neuts [25]. The performance measures of the system are calculated and the optimization problem to minimize the total expected cost rate is solved.

Suganya et al. [32] discussed a model of MAP/M/2/N/(s,S)/M/DFB with two heterogeneous servers (Server-1 and Server-2). An arriving p-customer gets service from Server-1 and Server-2 with different parameters and p-customers who occur during stock-out periods and/or during both servers' busy periods, wait in the finite waiting room. If the waiting room is full, then arriving p-customer considers to be lost. After the completion of service, the p-customer will decide either to join the orbit (infinite size) for additional service or leaves the system according to a Bernoulli trial. F-customers are served on the Server-2 and compete for their demands according to an exponential distribution. Multiple feedback is allowed. If Server-2 completes the service for the p-customer, then it either searches the f-customers in the orbit or becomes empty until the f-customer captures the server. After completing the service for f-customers, Server-2 is busy for the p-customer. The steady-state probabilities and the system performance measures are calculated by using MGM. Optimization problem to minimize the total expected cost rate is solved.

Models M/M/1/N/(s, S)/M/DFB and M/M/1/∞/(s, S)/M/DFB have been investigated by Melikov et al. [22]. The p-customer either joins the queue according to Bernoulli trial or leaves the system if the inventory level is zero at the moment of arrival. The customers in the queue become impatient when the inventory level drops to zero and they independently leave the system after waiting some exponentially distributed period of time. Authors considered the models both with finite and infinite size of orbits for f-customer. There are three options after the service completion of the p-customer: (i) customer leaves the system without purchasing an inventory item, (ii) customer purchases the item and leaves the system, (iii) customer does not purchase the item and joins the orbit for "decision making". The served f-customer may re-join the orbit, assuming the repetitive orbit joins are possible. The f-customers in orbit are assumed to be insistent, i.e. if the queue is full or the inventory level is zero at the moment of arrival, the f-customer returns to the orbit. The service time depends on whether the customer (primary or feedback) purchases the item or not, i.e. it has an exponential distribution with different parameters for each case. For calculation of the steady-state probabilities and the performance measures of the system, the space merging method is used. Total cost minimization problem is solved as well.

In all the papers considered above except Amirthakodi and Sivakumar [3], the models of QIS with DFB are investigated. Also, in all the models studied in the literature of queueing-inventory system with feedback schemes, it is assumed that, parameters of Bernoulli trial are constant, i.e. they are state-independent. It is obvious that, this assumption is usually unrealistic, since feedback of primary (or feedback) customers in general depend on the current inventory level of the system upon completing the service of a customer. The present work is close in spirit to Amirthakodi et al. [1]. However, the main differences between our model and the model considered in Amirthakodi et al. [1] are as follows:

(i) we consider the model with common waiting room for primary and feedback customers;

(ii) in our model, both kind of customers (primary and feedback) require inventory;

(iii) in our model, parameters of Bernoulli trial that determine feedback scheme are state-dependent;

(iv) in our model, customers are impatient in queue when the inventory level drops to zero.

On the one hand, taking into account the above assumptions (i)-(iv) increases the adequacy of the model to real situations; On the other hand, the study of such QISwFB models enriches the arsenal of operations research theory.

The rest of the paper is organized as follows. Section 2 provides the model assumptions and basic notations which are used throughout the paper. Infinitesimal matrix of the appropriate multi-dimensional Markov chain is derived in Section

3. In Section 4, the ergodicity condition is obtained and the method to calculate steady-state probabilities is developed. Desired system performance measures and long run total expected cost rate are calculated in Section 5. Section 6 provides results of numerical experiments. Also discussed the model with finite queue size in Section 7. Section 8 contains concluding remarks and direction of further researches.

The following abbreviations and notations are used in this manuscript:

QIS	queueing-inventory system
QS	queuing system
ICS	inventory control system
QISwFB	queueing-inventory systems with feedback
QISwDFB	queueing-inventory systems with delayed feedback
IFB	instantaneous feedback
DFB	delayed feedback
CTMC	continuous time Markov chain
LDQBD	level dependent quasi-birth and death process
MAP	Markovian Arrival Processes
PH	phase type distribution

p-customers	primary customers
f-customers	feedback customers
$\mathbf{e}$	column vector of 1's of appropriate order
$\mathbf{0}$	column vector of 0's of appropriate order
$\mathbf{I}$	Identity matrix of appropriate order
$\mathbf{O}$	Zero matrix of appropriate order
$A \otimes B$	For matrices $A_{m\times n}$ and $B_{p\times q}$ the Kronecker product of A and B is $A \otimes B = (a_{ij}B)$ of order $mp \times nq$
$C \oplus D$	The Kronecker sum of two square matrices C and D of order m and n respectively is $C \oplus D = C \oplus I_n + I_m \oplus D$
arrival process	$MAP(B_0, B_1)$ of order b
service time	$PH(\boldsymbol{\xi}, A)$ of order a
$\beta(i), 1 \leq i \leq S$	probability of customer leaves the system with current level inventory i
$1-\beta(i)$	probability of customer instantaneously joins the queue
τ	rate of customers in the queue become impatient when the inventory level is zero
η	rate of exponentially distributed lead time
α	probability of p-customer joins the queue
$1-\alpha$	probability of p-customer leaves the system

2. Model Assumptions

Pictorial presentation of the investigated system is shown in fig. 1. We consider model QIS of the type MAP/PH/1/•/(s, S)/M/IFB, in which all the inventory items are considered identical and after the service completion, the inventory level decreases by a single unit if the customer purchases the item (in fourth position symbol • is indicated, since we consider both models with finite and infinite queue sizes). Arrival of primary customers follows Markovian arrival process with representation (B_0, B_1) of order b. Let δ be the steady state probability vector of $B = B_0 + B_1$. Then, δ satisfy $\delta B = 0$ and $\delta \mathbf{e} = 1$. The fundamental rate of arrival is given by $\lambda = \delta B_1 \mathbf{e}$ which gives the expected number of arrivals per unit of time. If at the moment of the p-customer arrival inventory level is positive and the server is empty, then the p-customer is immediately taken to the service by the server; if the server is busy, then arrived p-customers join the queue independently on the inventory level. If the inventory level is zero at the arrival moment of p-customer, then according to Bernoulli trial, the p-customer either joins the queue with probability (w.p.) α or leaves the system w.p. $1-\alpha$. Service time of the p-customer has phase type distribution with representation $(\boldsymbol{\xi}, A)$ of order a. The transition rates within the set $\{1, 2, ..., a\}$ are defined by the generator A and the transition rates into the absorbing state are given by $\mathbf{A}^0 = -A\mathbf{e}$. The mean service of the customer is calculated by $\mu' = -\boldsymbol{\xi} A^{-1}\mathbf{e}$. At the end of each service the inventory level decreases by one unit.

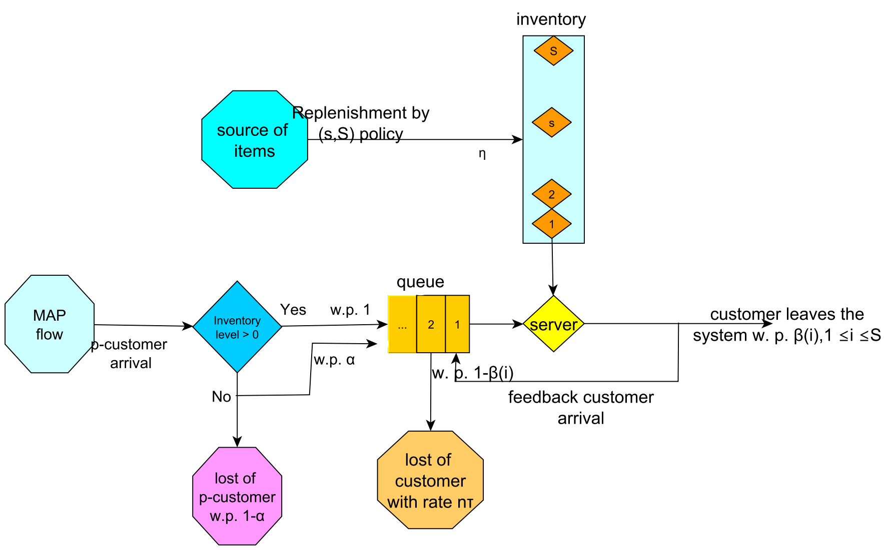


Figure 1: Picture presentation of the investigated system

It is assumed that, upon completion of the service, the customers have information related to current inventory level and feedback decision depends on the value of this quantity, i.e. there are two options after the service completion of the customer: (i) customer leaves the system w.p. $\beta(i)$, where i is current level of inventory, $i = 1, .., S$ (ii) customer instantaneously joins the queue w.p. $1 - \beta(i)$. If upon arrival of f-customer queue is empty and inventory level is positive, the server starts service of f-customer immediately; in other cases, an arrived f-customer joins the queue.

We assume that the system does not differentiate between p-customers and f-customers, i.e. service times of both kind of customers are the same. Every served customer (p-customer or f-customer) may re-join the queue as well, assuming the repetitive feedback is possible.

The customers in the queue become impatient when the inventory level is zero and they independently leave the system after waiting some exponentially distributed period of time with mean τ^{-1} i.e. customers leave the queue with total rate $n\tau$ when number of customers in queue is equal to $n, n \geq 1$.

In order to be short, here we assume that (s, S) replenishment policy is used and the lead time of the order is an exponentially distributed random variable with finite mean η (extending the results obtained here to other types of replenishment policies is not difficult).

The problem is to find the steady-state distribution of the system, determine the performance measures and perform the cost analysis of the system.

3. Appropriate multi-dimensional Markov chain

In this section, we discuss about appropriate Markov chain of the investigated queueing system with infinite capacity. Let $N(t)$ be the number of customers in the system at time t, $I(t)$ be the number of items inventory at time t, $C(t)$ be the phase of service process and $M(t)$ be the phase of arrival process. Then $\{(N(t), I(t), C(t), M(t)), t \geq 0\}$ is a CTMC which is a level dependent quasi-birth and death process (LDQBD) with the state space $\{(0, i, k), 1 \leq i \leq S, 1 \leq k \leq b\} \bigcup \{(n, 0, k), n \geq 0, 1 \leq k \leq b\} \bigcup \{(n, i, j, k), n \geq 1, 1 \leq i \leq S, 1 \leq j \leq a, 1 \leq k \leq b\}$. The infinitesimal generator of this CTMC is given by

$$\mathcal{W} = \begin{pmatrix} D_{10} & D_{00} & & & & \\ D_{21} & D_{11} & D_0 & & & \\ & D_{22} & D_{12} & D_0 & & \\ & & D_{23} & D_{13} & D_0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1}$$

where

$$D_{00} = \begin{pmatrix} \alpha B_1 & & & \\ & \boldsymbol{\xi} \otimes B_1 & & \\ & & \ddots & \\ & & & \boldsymbol{\xi} \otimes B_1 \end{pmatrix}, D_0 = \begin{pmatrix} \alpha B_1 & & & \\ & I \otimes B_1 & & \\ & & \ddots & \\ & & & I \otimes B_1 \end{pmatrix},$$

$$D_{21} = \begin{pmatrix} \tau I & & & \\ u_{10} & & & \\ & \ddots & & \\ & & u_{S0} & 0 \end{pmatrix}, D_{2n} = \begin{pmatrix} n\tau I & & & & \\ u_{10} & & & & \\ & u_2 & & & \\ & & \ddots & & \\ & & & u_S & 0 \end{pmatrix}, n \geq 2$$

$$D_{10} = \begin{pmatrix} v_0 & & & & & & \eta I \\ & v_1 & & & & & \eta I \\ & & \ddots & & & & \vdots \\ & & & v_1 & & & \eta I \\ & & & & B_0 & & \\ & & & & & \ddots & \\ & & & & & & B_0 \end{pmatrix},$$

$$D_{1n} = \begin{pmatrix} \hat{v}_0 & & & & & & \eta \boldsymbol{\xi} \otimes I \\ \hat{u}_1 & \hat{v}_1 & & & & & \eta I \\ & \ddots & \ddots & & & & \vdots \\ & & \hat{u}_s & \hat{v}_1 & & & \eta I \\ & & & \hat{u}_{s+1} & \hat{v}_2 & & \\ & & & & \ddots & \ddots & \\ & & & & & \hat{u}_S & \hat{v}_2 \end{pmatrix}, n \geq 1$$

with $u_{i0} = \beta(i)\mathbf{A}^0 \otimes I, 1 \leq i \leq S$;
$u_i = \beta(i)\mathbf{A}^0\boldsymbol{\xi} \otimes I, 2 \leq i \leq S$;
$v_0 = -(\alpha B_1 + \eta I); v_1 = B_0 - \eta I$
$\hat{v}_0 = -[\alpha B_1 + (\eta + n\tau)I]; \hat{v}_1 = A \oplus B_0 - \eta I; \hat{v}_2 = A \oplus B_0$
$\hat{u}_1 = [1 - \beta(1)]\mathbf{A}^0 \otimes I,$
$\hat{u}_i = [1 - \beta(i)]\mathbf{A}^0\boldsymbol{\xi} \otimes I, 2 \leq i \leq S.$

Note that the matrices $D_0, D_{1n}; n \geq 1$ and $D_{2n}; n \geq 2$ are of order q and matrices D_{10}, D_{00}, D_{21} are of order $p \times p, p \times q, q \times p$ respectively where $p = (S+1)b, q = (Sa+1)b$.

4. Steady-state probability vector

Let $\mathbf{w}$, partitioned as $\mathbf{w} = \{\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, ...\}$ be the limiting probability vector of $\mathcal{W}$. Then

$$\mathbf{w}\mathcal{W} = 0, \mathbf{w}\mathbf{e} = 1. \tag{2}$$

From (2) we have

$$\begin{aligned}
\mathbf{w}_0 D_{10} + \mathbf{w}_1 D_{21} &= \mathbf{0}, \\
\mathbf{w}_0 D_{00} + \mathbf{w}_1 D_{11} + \mathbf{w}_2 D_{22} &= \mathbf{0}, \\
\mathbf{w}_{n-1} D_0 + \mathbf{w}_n D_{1n} + \mathbf{w}_{n+1} D_{2n+1} &= \mathbf{0}, n \geq 2
\end{aligned} \tag{3}$$

subject to the normalizing condition

$$\sum_{n=0}^{\infty} \mathbf{w}_n \mathbf{e} = 1. \tag{4}$$

4.1. Neuts–Rao Truncation Method

We derive the steady state probability vector $\mathbf{w}$ of the generator $\mathcal{W}$ by Neuts–Rao Truncation Method (see [26]). For this identifying a positive integer,say, N such that defining $D_1 = D_{1n}; D_2 = D_{2n}$ for $n \geq N$ we use the following generator obtained from (1) to describe our model under study.

$$\tilde{\mathcal{W}} = \begin{pmatrix}
D_{10} & D_{00} & & & & & \\
D_{21} & D_{11} & D_0 & & & & \\
 & \ddots & \ddots & \ddots & & & \\
 & & D_{2N-1} & D_{1N-1} & D_0 & & \\
 & & & D_2 & D_1 & D_0 & \\
 & & & & \ddots & \ddots & \ddots
\end{pmatrix} \tag{5}$$

4.1.1. Stability Condition

Let $\boldsymbol{\phi}$ be the steady state probability vector of

$$D = D_0 + D_1 + D_2 = \begin{pmatrix} -\eta I & & & & & & & \eta\boldsymbol{\xi}\otimes I \\ H_0 & G_s & & & & & & \eta I \\ & H & G_s & & & & & \eta I \\ & & \ddots & \ddots & & & & \vdots \\ & & & H & G_s & & & \eta I \\ & & & & H & G & & \\ & & & & & \ddots & \ddots & \\ & & & & & & H & G \end{pmatrix}$$

with $H_0 = \mathbf{A}^0 \otimes I, H = \mathbf{A}^0\boldsymbol{\xi} \otimes I, G = A \oplus B, G_s = A \oplus B - \eta I$.

Then

$$\boldsymbol{\phi} D = 0 \text{ and } \boldsymbol{\phi}\mathbf{e} = 1. \tag{6}$$

The components of $\boldsymbol{\phi}$ are obtained as where

$$\boldsymbol{\phi}_i = \begin{cases} \frac{1}{\eta}\boldsymbol{\phi}_S \mathcal{K}^s K_s^{S-(s+1)} H_0, i = 0; \\ \boldsymbol{\phi}_S \mathcal{K}^s K_s^{S-(i-1)}, 1 \le i \le s; \\ \boldsymbol{\phi}_S \mathcal{K}^{s-(i-(s+1))}, s+1 \le i \le S-1 \end{cases} \tag{7}$$

where $\mathcal{K} = -H(G^{-1}), K_s = -H(G_s^{-1})$.

The unknown probability $\boldsymbol{\phi}_S$ can be found from the normalizing condition $\boldsymbol{\phi}\mathbf{e} = 1$ as

$$\boldsymbol{\phi}_S \left[\frac{1}{\eta}\mathcal{K}K_s^{S-(s+1)} H_0 + \mathcal{K}^s \sum_{i=1}^{s} K_s^i + \sum_{i=1}^{s} \mathcal{K}^i + I \right] \mathbf{e} = 1. \tag{8}$$

The following theorem provides the stability condition of the system under study.

Theorem 1. *The queueing-inventory system under study is stable if and only if*

$$\boldsymbol{\phi}_S \left(\mathcal{U}\alpha B_1 + \mathcal{A}(I \otimes B_1) \right) \boldsymbol{e} < \boldsymbol{\phi}_S \left(N\tau\mathcal{U} + \mathcal{B}_1(\boldsymbol{A}^0 \otimes I) + \sum_{i=2}^{S} \mathcal{B}_i(\boldsymbol{A}^0\boldsymbol{\xi} \otimes I) \right) \boldsymbol{e} \tag{9}$$

where $\mathcal{U} = \frac{1}{\eta}\mathcal{K}^s K_s^{S-(s+1)} H_0, \mathcal{A} = \mathcal{K}^s \sum_{i=1}^{s} K_s^i + \sum_{i=1}^{s} \mathcal{K}^i + I,$

$$\mathcal{B}_i = \begin{cases} \beta(1)\mathcal{K}^s K_s^s, & i = 1; \\ \mathcal{K}^s K_s^{s-(i-1)}\beta(i), & 2 \le i \le s; \\ \mathcal{K}^{s-(i-(s+1))}\beta(i), & s+1 \le i \le S-1; \\ \beta(S), & i = S. \end{cases}.$$

Proof. The queueing-inventory system under study with the LIQBD process with generator given in (5) is stable if and only if (see Neuts [25])

$$\boldsymbol{\phi} D_0 \mathbf{e} < \boldsymbol{\phi} D_2 \mathbf{e}. \tag{10}$$

From the matrices D_0 and D_2 we get

$$\boldsymbol{\phi} D_0 \mathbf{e} = \boldsymbol{\phi}_0 \alpha B_1 \mathbf{e} + \sum_{i=1}^{S} \boldsymbol{\phi}_i (I \otimes B_1)\mathbf{e}$$

and

$$\boldsymbol{\phi} D_2 \mathbf{e} = N\tau \boldsymbol{\phi}_0 \mathbf{e} + \boldsymbol{\phi}_1 \beta(1)(\mathbf{A}^0 \otimes I)\mathbf{e} + \sum_{I=2}^{s} \boldsymbol{\phi}_i \beta(i)(\mathbf{A}^0 \boldsymbol{\xi} \otimes I)\mathbf{e}.$$

Using (7) we obtain the stated result given in (9).

Now let $\tilde{\mathbf{w}}$ be the steady-state probability vector of $\tilde{\mathcal{W}}$. Then

$$\tilde{\mathbf{w}}\tilde{\mathcal{W}} = \mathbf{0} \text{ and } \tilde{\mathbf{w}}\mathbf{e} = 1. \tag{11}$$

By Neuts [25], the vectors $\tilde{\mathbf{w}}_n, n \ge N$ are

$$\tilde{\mathbf{w}}_{n+N-1} = \tilde{\mathbf{w}}_{N-1}\mathcal{Q}^n, \text{ for } n \ge 1, \tag{12}$$

where $\mathcal{Q}$ is the minimal non-negative solution of the matrix quadratic equation

$$\mathcal{Q}^2 D_2 + \mathcal{Q} D_1 + D_0 = 0. \tag{13}$$

The components $\tilde{\mathbf{w}}_n, 0 \le n \le N-1$ are obtained by solving the following set of equations

$$\begin{aligned} \tilde{\mathbf{w}}_0 D_{10} + \tilde{\mathbf{w}}_1 D_{21} &= \mathbf{0}, \\ \tilde{\mathbf{w}}_0 D_{00} + \tilde{\mathbf{w}}_1 D_{11} + \tilde{\mathbf{w}}_2 D_{22} &= \mathbf{0}, \\ \tilde{\mathbf{w}}_{n-1} D_0 + \tilde{\mathbf{w}}_n D_{1n} + \tilde{\mathbf{w}}_{n+1} D_{2n+1} &= \mathbf{0}, 2 \le n \le N-2 \end{aligned} \tag{14}$$

$$\tilde{\mathbf{w}}_{N-2}D_0 + \tilde{\mathbf{w}}_{N-1}[D_{1N-1} + \mathcal{Q}D_2] = \mathbf{0}$$

and the normalizing condition

$$\sum_{n=0}^{N-2} \tilde{\mathbf{w}}_n\mathbf{e} + \tilde{\mathbf{w}}_{N-1}(I - \mathcal{Q})^{-1}\mathbf{e} = 1 \tag{15}$$

where $\tilde{\mathbf{w}}_n = \tilde{\mathbf{w}}_0 \prod_{i=1}^{n} \mathcal{H}_i, 1 \leq n \leq N-1$ with

$$\mathcal{H}_i = \begin{cases} -D_{00}[D_{11} + \mathcal{H}_2 D_{22}]^{-1}, & i = 1 \\ -D_0[D_{1i} + \mathcal{H}_{i+1}D_{2i+1}]^{-1}, & 2 \leq i \leq N-2 \\ -D_0[D_{1N-1} + \mathcal{Q}D_2]^{-1}, & i = N-1. \end{cases} \tag{16}$$

5. Some Important performance measures

In this section, we list a few system performance measures along with their formulae, to bring out the qualitative nature of the model under study.

1. Probability that the server is idle, $p_0 = \left[\sum_{i=1}^{S} \tilde{\mathbf{w}}_0(i) + \sum_{n=0}^{\infty} \tilde{\mathbf{w}}_n(0)\right]\mathbf{e}$
2. Mean number of customers in the system, $E_N = \sum_{n=1}^{\infty} n\tilde{\mathbf{w}}_n\mathbf{e}$
3. Mean number of items in the inventory, $E_I = \left[\sum_{i=1}^{S} i\tilde{\mathbf{w}}_0(i) + \sum_{n=1}^{\infty}\sum_{i=1}^{S} i\tilde{\mathbf{w}}_n(i)\right]\mathbf{e}$
4. Expected purchase rate, $E_P = \frac{1}{\mu'}\sum_{n=1}^{\infty}\sum_{i=1}^{S} \tilde{\mathbf{w}}_n(i)\mathbf{e}$
5. Expected rate of feedback, $E_F = \frac{1}{\mu'}\sum_{n=1}^{\infty}\sum_{i=1}^{S}(1-\beta(i))\tilde{\mathbf{w}}_n(i)\mathbf{e}$
6. Expected rate of customers leaves the system after service completion (no feedback), $E_{NF} = \frac{1}{\mu'}\sum_{n=1}^{\infty}\sum_{i=1}^{S}\beta(i)\tilde{\mathbf{w}}_n(i)\mathbf{e}$

7. Expected loss rate of customers due to no items in inventory,
$E_L = (1-\alpha)\lambda \sum_{n=0}^{\infty} \tilde{\mathbf{w}}_n(0)\mathbf{e}$
8. The rate of abandonment of the system due to customer impatience when no item in the inventory, $E_{imp} = \sum_{n=1}^{\infty} n\tau \tilde{\mathbf{w}}_n(0)\mathbf{e}$
9. Expected rate of replenishment,
$E_R = \eta \left[\sum_{i=1}^{s} \tilde{\mathbf{w}}_0(i) + \sum_{n=0}^{\infty} \tilde{\mathbf{w}}_n(0) + \sum_{n=1}^{\infty} \sum_{i=1}^{s} \tilde{\mathbf{w}}_n(i) \right] \mathbf{e}$

6. Numerical experiments

Next we proceed to a few numerical examples in order to bring out the system behaviour with respect to certain parameters.

We assume that $N = 100$. PH service process of customer is characterized by

$$\boldsymbol{\xi} = \begin{pmatrix} 1 & 0 \end{pmatrix}, A = \begin{pmatrix} -6 & 6 \\ 0 & -6 \end{pmatrix}$$

for which the mean service time $\mu' = 0.3333$.

For the arrival process, we consider the following two sets of values for B_0 and B_1 as follows.

1. **MAP with positive correlation (P_{MAP}):**

$$B_0 = \begin{pmatrix} -1.25 & 1.25 & 0 \\ 0 & -1.25 & 0 \\ 0 & 0 & -4.5 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1.2375 & 0 & 0.0125 \\ 0.025 & 0 & 4.475 \end{pmatrix}$$

2. **MAP with negative correlation (N_{MAP}):**

$$B_0 = \begin{pmatrix} -1.25 & 1.25 & 0 \\ 0 & -1.25 & 0 \\ 0 & 0 & -4.5 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0.0125 & 0 & 1.2375 \\ 4.475 & 0 & 0.025 \end{pmatrix}$$

The above MAP processes will be normalized so as to have a specific arrival rate. However, these are qualitatively different in that they have different correlation structure. The arrival processes labeled N_{MAP} has negative correlation with value -0.4640 and P_{MAP} has positive correlation with value 0.4640 for two successive inter-arrival times.

Here probability of customer leaves the system with current inventory level i with probability $\beta(i), 1 \leq i \leq S$ and $\beta(1) \geq \beta(2) \geq \ldots \geq \beta(S)$. We use $\beta(i) = \beta(S-(i-1))^{\gamma}, \ 1 \leq i \leq K$ where $0 \leq \gamma \leq 1$. Choose $\gamma = 0.1$.

	P_{MAP}					
η	E_N	E_I	E_P	E_F	E_{imp}	E_R
1	14.9237	17.0202	1.3234	0.0272	0.0681	0.0789
1.5	18.5969	17.2739	1.3710	0.0328	0.0176	0.0847
2	19.8092	17.4102	1.3850	0.0359	0.0046	0.0874
2.5	20.1963	17.4942	1.3895	0.0378	0.0013	0.0889
3	20.3410	17.5509	1.3912	0.0391	0.0004	0.0898
	N_{MAP}					
η	E_N	E_I	E_P	E_F	E_{imp}	E_R
1	0.5585	17.1827Z	1.1271	0.0300	0.0003	0.0705
1.5	0.5601	17.4010	1.1295	0.0323	0.0002	0.0722
2	0.5608	17.5086	1.1307	0.0334	0.0001	0.0730
2.5	0.5613	17.5725	1.1314	0.0342	0.0001	0.0736
3	0.5616	17.6148	1.1319	0.0347	0.0000	0.0739

Table 1: Effect of η: fix $S = 25, s = 10, \tau = 3, \alpha = 0.7, \beta = 0.8$

	P_{MAP}						
s	E_N	E_I	E_P	E_F	E_L	E_{imp}	E_R
6	16.5614	10.4861	1.3711	0.0885	0.0005	0.0700	0.1393
7	18.6575	10.9840	1.4011	0.1012	0.0003	0.0463	0.1581
8	20.3731	11.4808	1.4258	0.1150	0.0002	0.0305	0.1811
9	21.7734	11.9749	1.4470	0.1301	0.0002	0.0202	0.2103
10	22.9519	12.4640	1.4663	0.1468	0.0001	0.0135	0.2491
11	23.9983	12.9448	1.4852	0.1655	0.0001	0.0093	0.3038
	N_{MAP}						
s	E_N	E_I	E_P	E_F	E_L	E_{imp}	E_R
6	0.5979	10.6006	1.1791	0.0812	0.0002	0.0007	0.1238
7	0.6067	11.1354	1.1902	0.0921	0.0001	0.0005	0.1397
8	0.6159	11.6112	1.2020	0.1035	0.0001	0.0004	0.1591
9	0.6273	12.1421	1.2158	0.1171	0.0000	0.0003	0.1858
10	0.6393	12.5997	1.2306	0.1314	0.0000	0.0002	0.2204
11	0.6549	13.1254	1.2488	0.1492	0.0000	0.0001	0.2732

Table 2: Effect of s: fix $S = 15, \eta = 2, \tau = 3, \alpha = 0.7, \beta = 0.8$

	P_{MAP}					
S	E_N	E_I	E_P	E_F	E_{imp}	E_R
16	22.5786	12.9714	1.4551	0.1306	0.0115	0.2114
17	22.2112	13.4731	1.4448	0.1162	0.0100	0.1833
18	21.8568	13.9711	1.4354	0.1031	0.0088	0.1617
19	21.5183	14.4665	1.4266	0.0912	0.0079	0.1445
20	21.1962	14.9598	1.4185	0.0802	0.0071	0.1305
21	20.8901	15.4518	1.4110	0.0701	0.0064	0.1189
	N_{MAP}					
S	E_N	E_I	E_P	E_F	E_{imp}	E_R
16	0.6273	13.1421	1.2158	0.1171	0.0003	0.1858
17	0.6159	13.6112	1.2020	0.1035	0.0002	0.1591
18	0.6067	14.1353	1.1903	0.0921	0.0001	0.1397
19	0.5979	14.6005	1.1792	0.0812	0.0001	0.1238
20	0.5905	15.1155	1.1696	0.0719	0.0000	0.1114
21	0.5833	15.5771	1.1604	0.0628	0.0000	0.1008

Table 3: Effect of S: fix $s = 10, \eta = 2, \tau = 3, \alpha = 0.7, \beta = 0.8$

	P_{MAP}						
β	p_0	E_N	E_I	E_P	E_F	E_{imp}	E_R
0.1	0.0002	172.0247	6.2748	1.1031	2.2684	0.0257	0.0679
0.2	0.0222	182.1024	16.8307	2.8904	2.1843	0.0253	0.1785
0.3	0.1554	87.9533	17.1806	2.5305	1.6039	0.0150	0.1573
0.4	0.2753	60.8206	17.2870	2.1733	1.1130	0.0112	0.1358
0.5	0.3647	45.6220	17.3439	1.9057	0.7442	0.0089	0.1195
0.6	0.4339	34.9211	17.3764	1.6981	0.4567	0.0072	0.1068
0.7	0.4903	26.6029	17.3967	1.5290	0.2254	0.0058	0.0963
0.8	0.5383	19.8092	17.4102	1.3850	0.0359	0.0046	0.0874
0.9	0.5806	14.1099	17.4200	1.2582	0.0112	0.0036	0.0795
1	0.6183	9.3648	17.4274	1.1450	0.0009	0.0026	0.0725
	N_{MAP}						
β	p_0	E_N	E_I	E_P	E_F	E_{imp}	E_R
0.1	0.0005	216.8600	8.2959	1.4579	1.2798	0.0339	0.0896
0.2	0.0011	295.1081	13.0886	2.2991	1.7374	0.0412	0.1412
0.3	0.0181	33.8037	17.0732	2.9458	1.8663	0.0111	0.1811
0.4	0.2492	2.5583	17.3045	2.2523	1.1538	0.0054	0.1402
0.5	0.3985	1.3257	17.4257	1.8045	0.7061	0.0009	0.1136
0.6	0.4982	0.9059	17.4852	1.5053	0.4073	0.0004	0.0957
0.7	0.5696	0.6918	17.5120	1.2913	0.1938	0.0002	0.0828
0.8	0.6231	0.5608	17.5086	1.1307	0.0334	0.0001	0.0730
0.9	0.6649	0.4715	17.4253	1.0052	0.0120	0.0000	0.0650
1	0.6983	0.4070	17.4069	0.9050	0.0017	0.0000	0.0587

Table 4: Effect of β: fix $s = 10, S = 25, \eta = 2, \tau = 3, \alpha = 0.7$

A quick look at Table 1, 2, 3 and 4 reveal some interesting observations.

- See Table 1
  - As η is increased, as expected, E_{imp} decreases for both arrival processes. However, the rate of decrease is much higher for N_{MAP} but P_{MAP} indicating the role of (positive) correlated arrivals.
  - While E_I, E_F, E_R increase as η increases, as is to be expected, the rate of increase is pretty much the same for both arrival processes.
  - As η is increased, E_N increases for both arrival processes. However, the rate of increase is much higher for P_{MAP}.

- From Table 2
  - We notice that the measures, E_I, E_F, E_P and E_R, increase as s increases for both positive and negative correlation.
  - As s increases, E_N increases for both arrival processes. However, much higher rate E_N in P_{MAP}.
  - While E_L and E_{imp} decrease as s increases, as is to be expected, the rate of decrease of E_L is pretty much the same for both arrival processes. However, the rate of decrease is much higher for P_{MAP}.
- Table 3 shows that
  - As S increases, E_N increases for both arrival processes. However, much higher rate E_N in positive correlation.
  - We notice that the measures, E_P, E_F, E_R and E_{imp}, decrease as S increases for both positive and negative correlation.
  - As S is increased, E_I increases for both arrival processes in similar way.
- Table 4 indicates that
  - p_0, E_I increase as β increases for both arrivals, as expected line. However, E_N, E_F, E_{imp} decrease.
  - As β increases, E_P, E_R first increases then decreases.

6.1. Revenue Function

Based on performance measures we define the revenue (profit) function of infinite case as follows:

$$F(\beta, s, S) = C_P E_P + C_F E_F - [C_I E_I + C_N E_N + C_L E_L + [\mathbf{F} + (S-s)C_R]E_R + C_{imp}E_{imp}]$$

where

C_P	revenue to the system due to per unit purchase by a customer after completion of his service
C_F	revenue due to feedback of customer
C_I	holding cost per inventoried item in the system per unit time
C_N	holding cost per customer in the infinite queue
C_L	cost due to customer lost per unit time
$\mathbf{F}$	fixed cost of delivery service
C_R	carriage cost of delivery service per item
C_{imp}	cost due to impatience of customer per unit time

In order to study the effect of different parameters on profit function we first take the values $(C_P, C_F, C_I, C_N, C_L, \mathbf{F}, C_R, C_{imp}) = (\$125, \$5, \$2, \$1, \$15, \$115, \$7, \$3)$.

We assign the following values to the parameters: $S = 25, s = 10, \alpha = 0.7, \eta = 2, \tau = 3, \beta = 0.8$. For different values of S and s the expected profit is calculated and presented in Table 5 (see figures 2). This table shows that the profit function first increases and then decreases in both case. Maximum revenue of the system of each cases marked as bold case.

	P_{MAP}				N_{MAP}		
S	$s = 10$	$s = 11$	$s = 12$	S	$s = 10$	$s = 11$	$s = 12$
17	101.9256	98.8262	94.7604	15	95.5882	90.8642	84.1191
18	102.4580	99.9617	96.8392	16	96.4661	93.5884	88.8643
19	**102.5862**	100.4966	97.9783	17	**96.8238**	94.4663	91.5884
20	102.4310	**100.6258**	98.5151	18	96.4702	**94.8239**	92.4663
21	102.0723	100.4708	**98.6452**	19	95.9797	94.4702	**92.8239**
22	101.5609	100.1117	98.4906	20	95.1274	93.9797	92.4702
23	100.9335	99.5997	98.1315	21	94.2725	93.9797	91.9797
	P_{MAP}				N_{MAP}		
s	$S = 15$	$S = 20$	$S = 25$	s	$S = 15$	$S = 20$	$S = 25$
3	108.9762	106.3343	102.3850	1	106.8733	101.5580	95.7669
4	109.8202	107.2832	103.1977	2	**107.5024**	**101.7656**	**95.8154**
5	**109.9591**	**107.5559**	**103.3710**	3	106.9592	101.3194	95.0968
6	109.2793	107.2191	103.0444	4	106.1966	100.5575	94.5176
7	107.8324	106.4348	102.3882	5	105.0994	99.8438	93.6620
8	105.6673	105.3359	101.5255	6	103.9698	98.9601	93.0009
9	102.7193	103.9944	100.5283	7	102.4663	98.1621	92.1482

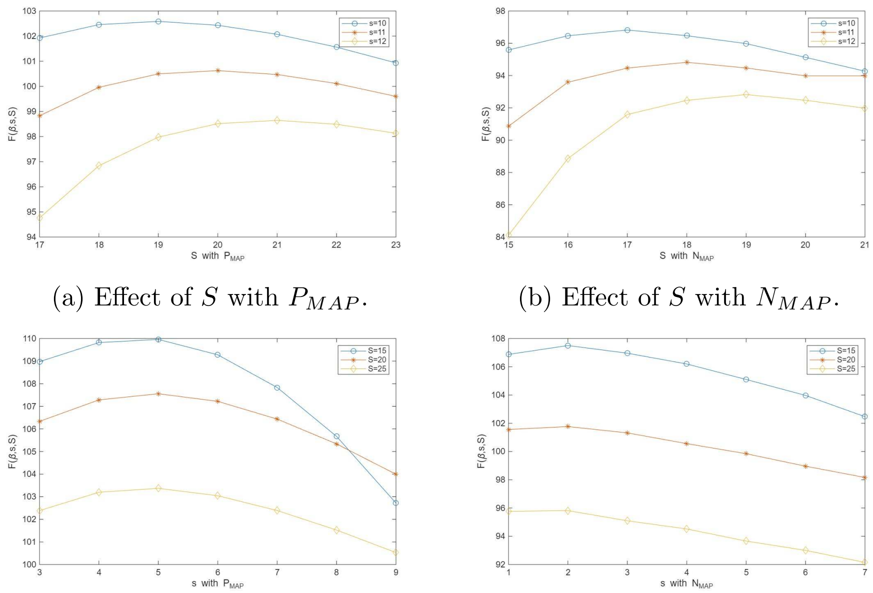
Table 5: Effect of s and S

Table 6 represents the revenue to the system as a function of β for MAP with positive and negative correlations. In both cases the cost function behaves as a concave function (see Fig. 3).

7. Model with finite queue size

Next we consider a finite case of the model described in Sec. 2. This is of particular interest since the system is always stable.

In this section, we consider the system having finite waiting space, say K. Other assumptions remain the same as considered above in Sec. 2.



(a) Effect of S with P_{MAP}.

(b) Effect of S with N_{MAP}.

(c) Effect of s with P_{MAP}.

(d) Effect of S with N_{MAP}.

Figure 2: Effect of s and S on $F(\beta, s, S)$

β	P_{MAP}	N_{MAP}
0.1	-56.8451	-64.6281
0.2	117.1141	-56.4033
0.3	**167.3740**	**269.7286**
0.4	151.9308	219.2900
0.5	135.3027	167.9145
0.6	121.3633	133.2594
0.7	109.6412	108.4459
0.8	99.4329	89.8568
0.9	90.2207	75.5670
1	81.7070	64.0211

Table 6: Effect of β on $F(\beta, s, S)$

Let $N_f(t)$ be the number of customers in the finite system with capacity K at time t. Then the process $\{(N_f(t), I(t), C(t), M(t)), t \geq 0\}$ is a CTMC with state space given by $\{(0, i, k), 1 \leq i \leq S, 1 \leq k \leq b\} \bigcup \{(n, 0, k), 0 \leq n \leq K, 1 \leq k \leq b\} \bigcup \{(n, i, j, k), 1 \leq n \leq K, 1 \leq i \leq S, 1 \leq j \leq a, 1 \leq k \leq b\}$. The infinitesimal

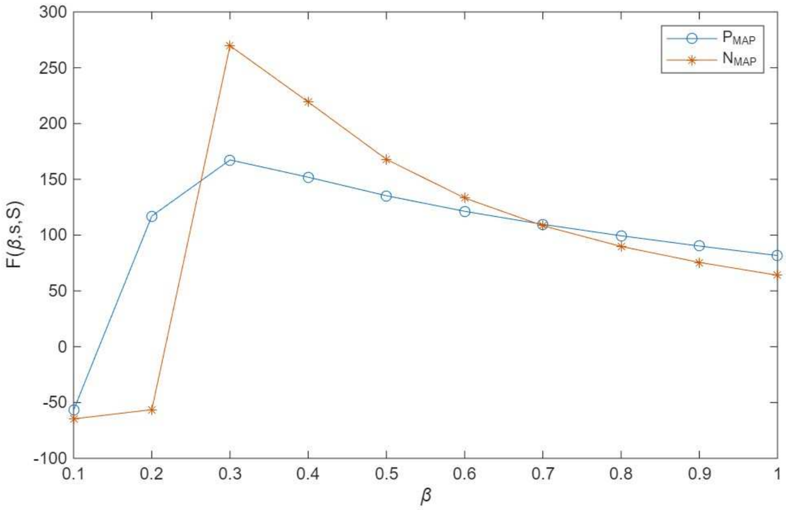


Figure 3: Effect of β on $F(\beta, s, S)$

generator of this CTMC is given by

$$\hat{\mathcal{W}} = \begin{pmatrix} D_{10} & D_{00} & & & & \\ D_{21} & D_{11} & D_0 & & & \\ & D_{22} & D_{12} & D_0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & D_{2K-1} & D_{1K-1} & D_0 \\ & & & & D_{2K} & D_{1K} \end{pmatrix} \tag{17}$$

where

$$D_{00} = \begin{pmatrix} \alpha B_1 & & & \\ & \boldsymbol{\xi} \otimes B_1 & & \\ & & \ddots & \\ & & & \boldsymbol{\xi} \otimes B_1 \end{pmatrix}, D_0 = \begin{pmatrix} \alpha B_1 & & & \\ & I \otimes B_1 & & \\ & & \ddots & \\ & & & I \otimes B_1 \end{pmatrix},$$

$$D_{21} = \begin{pmatrix} \tau I & & & \\ u_{10} & & & \\ & \ddots & & \\ & & u_{S0} & 0 \end{pmatrix}, D_{2n} = \begin{pmatrix} n\tau I & & & & \\ u_{10} & & & & \\ & u_2 & & & \\ & & \ddots & & \\ & & & u_S & 0 \end{pmatrix}, 2 \le n \le K$$

$$D_{10} = \begin{pmatrix} v_0 & & & & & & \eta I \\ & v_1 & & & & & \eta I \\ & & \ddots & & & & \vdots \\ & & & v_1 & & & \eta I \\ & & & & B_0 & & \\ & & & & & \ddots & \\ & & & & & & B_0 \end{pmatrix}, D_{1K} = \begin{pmatrix} \hat{v'}_0 & & & & & & & \eta\boldsymbol{\xi} \otimes I \\ \hat{u}_1 & \hat{v'}_1 & & & & & & \eta I \\ & \ddots & \ddots & & & & & \vdots \\ & & \hat{u}_s & \hat{v'}_1 & & & & \eta I \\ & & & \hat{u}_{s+1} & \hat{v'}_2 & & & \\ & & & & \ddots & \ddots & & \\ & & & & & \hat{u}_S & & \hat{v'}_2 \end{pmatrix},$$

$$D_{1n} = \begin{pmatrix} \hat{v}_0 & & & & & & & \eta\boldsymbol{\xi} \otimes I \\ \hat{u}_1 & \hat{v}_1 & & & & & & \eta I \\ & \ddots & \ddots & & & & & \vdots \\ & & \hat{u}_s & \hat{v}_1 & & & & \eta I \\ & & & \hat{u}_{s+1} & \hat{v}_2 & & & \\ & & & & \ddots & \ddots & & \\ & & & & & \hat{u}_S & & \hat{v}_2 \end{pmatrix}, 1 \leq n \leq K-1$$

with $u_{i0} = \beta(i)\mathbf{A}^0 \otimes I, 1 \leq i \leq S$;
$u_i = \beta(i)\mathbf{A}^0\boldsymbol{\xi} \otimes I, 2 \leq i \leq S$;
$v_0 = -(\alpha B_1 + \eta I); v_1 = B_0 - \eta I$
$\hat{v}_0 = -[\alpha B_1 + (\eta + n\tau)I]; \hat{v}_1 = A \oplus B_0 - \eta I; \hat{v}_2 = A \oplus B_0$
$\hat{v'}_0 = B - (\eta + n\tau)I; \hat{v'}_1 = A \oplus B - \eta I; \hat{v'}_2 = A \oplus B$
$\hat{u}_1 = [1 - \beta(1)]\mathbf{A}^0 \otimes I,$
$\hat{u}_i = [1 - \beta(i)]\mathbf{A}^0\boldsymbol{\xi} \otimes I, 2 \leq i \leq S.$

Note that the matrices $D_0, D_{1n}; 1 \leq n \leq K$ and $D_{2n}; 2 \leq n \leq K-1$ are of order q and matrices D_{10}, D_{00}, D_{21} are of order $p \times p, p \times q, q \times p$ respectively where $p = (S+1)b, q = (Sa+1)b$.

Let $\hat{\mathbf{w}}$, partitioned as $\hat{\mathbf{w}} = \{\hat{\mathbf{w}}_0, \hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, ..., \hat{\mathbf{w}}_K\}$ be the steady state probability vector of $\hat{\mathcal{W}}$. Then

$$\hat{\mathbf{w}}\hat{\mathcal{W}} = \mathbf{0}, \hat{\mathbf{w}}\mathbf{e} = 1. \tag{18}$$

From (18) we have

$$\begin{aligned} \hat{\mathbf{w}}_0 D_{10} + \hat{\mathbf{w}}_1 D_{21} &= \mathbf{0}, \\ \hat{\mathbf{w}}_0 D_{00} + \hat{\mathbf{w}}_1 D_{11} + \hat{\mathbf{w}}_2 D_{22} &= \mathbf{0}, \\ \hat{\mathbf{w}}_{n-1} D_0 + \hat{\mathbf{w}}_n D_{1n} + \hat{\mathbf{w}}_{n+1} D_{2n+1} &= \mathbf{0}, 2 \leq n \leq K-1 \\ \hat{\mathbf{w}}_{K-1} D_0 + \hat{\mathbf{w}}_K D_{1K} &= \mathbf{0} \end{aligned} \tag{19}$$

subject to the normalizing condition

$$\sum_{n=0}^{K} \hat{\mathbf{w}}_n \mathbf{e} = 1. \tag{20}$$

Solving the above set of equations we get $\hat{\mathbf{w}}_n = \hat{\mathbf{w}}_0 \prod_{i=1}^{n} \mathcal{G}_i, 1 \leq i \leq K$ where

$$\mathcal{G}_i = \begin{cases} -D_{00}[D_{11} + \mathcal{G}_2 D_{22}]^{-1}, & i = 1 \\ -D_0[D_{1i} + \mathcal{G}_{i+1} D_{2i+1}]^{-1}, & 2 \leq i \leq K-1 \\ -D_0 D_{1K}^{-1}, \quad i = K \end{cases} \tag{21}$$

and from the normalizing condition given in (20) we have

$$\hat{\mathbf{w}}_0 \left[I + \sum_{n=1}^{K} \prod_{i=1}^{n} \mathcal{G}_i \right] \mathbf{e} = 1. \tag{22}$$

7.1. System performance measures

1. Probability that the server is idle, $p_0^{(f)} = \left[\sum_{i=1}^{S} \hat{\mathbf{w}}_0(i) + \sum_{n=0}^{K} \hat{\mathbf{w}}_n(0) \right] \mathbf{e}$
2. Mean number of customers in the system, $E_N^{(f)} = \sum_{n=1}^{K} n \hat{\mathbf{w}}_n \mathbf{e}$
3. Mean number of items in the inventory, $E_I^{(f)} = \left[\sum_{i=1}^{S} i \hat{\mathbf{w}}_0(i) + \sum_{n=1}^{K} \sum_{i=1}^{S} i \hat{\mathbf{w}}_n(i) \right] \mathbf{e}$
4. Expected purchase rate, $E_P^{(f)} = \frac{1}{\mu'} \sum_{n=1}^{K} \sum_{i=1}^{S} \hat{\mathbf{w}}_n(i) \mathbf{e}$
5. Expected rate of feedback $E_F^{(f)} = \frac{1}{\mu'} \sum_{n=1}^{K} \sum_{i=1}^{S} (1 - \beta(i)) \hat{\mathbf{w}}_n(i) \mathbf{e}$
6. Expected rate of customers leaves the system after service completion (no feedback), $E_{NF}^{(f)} = \frac{1}{\mu'} \sum_{n=1}^{K} \sum_{i=1}^{S} \beta(i) \hat{\mathbf{w}}_n(i) \mathbf{e}$

7. Expected loss rate of customers due to no items in the system, $E_L^{(f)} = (1-\alpha)\lambda \sum_{n=0}^{K-1} \hat{\mathbf{w}}_n(0)\mathbf{e}$

8. The rate of abandonment of the system due to customer impatience when no item in the inventory, $E_{imp}^{(f)} = \sum_{n=1}^{K} n\tau\hat{\mathbf{w}}_n(0)\mathbf{e}$

9. Expected rate of replenishment, $E_R^{(f)} = \eta \left[\sum_{i=1}^{s} \hat{\mathbf{w}}_0(i) + \sum_{n=0}^{K} \hat{\mathbf{w}}_n(0) + \sum_{n=1}^{K}\sum_{i=1}^{s} \hat{\mathbf{w}}_n(i)\right]\mathbf{e}$

7.2. Numerical experiments

In this section we consider the numerical examples with the finite waiting space $K = 200$. Other parameters remain the same as considered above in Sec. 6.

	$S = 15$ with P_{MAP}						
s	$E_N^{(f)}$	$E_I^{(f)}$	$E_P^{(f)}$	$E_F^{(f)}$	$E_L^{(f)}$	$E_{imp}^{(f)}$	$E_R^{(f)}$
6	18.1045	10.4822	1.3847	0.0893	0.0006	0.0983	0.1404
7	20.5023	10.9780	1.4221	0.1025	0.0004	0.0688	0.1601
8	22.4191	11.4728	1.4535	0.1169	0.0002	0.0472	0.1840
9	23.9566	11.9652	1.4805	0.1327	0.0002	0.0321	0.2143
10	25.2377	12.4527	1.5048	0.1501	0.0001	0.0219	0.2544
	$s = 10$ with P_{MAP}						
S	$E_N^{(f)}$	$E_I^{(f)}$	$E_P^{(f)}$	$E_F^{(f)}$	$E_L^{(f)}$	$E_{imp}^{(f)}$	$E_R^{(f)}$
15	25.2377	12.4527	1.5048	0.1501	0.0001	0.0219	0.2544
16	24.8206	12.9609	1.4924	0.1336	0.0001	0.0186	0.2157
17	24.4115	13.4635	1.4809	0.1187	0.0001	0.0160	0.1870
18	24.0177	13.9621	1.4703	0.1052	0.0001	0.0141	0.1649
19	23.6420	14.4580	1.4604	0.0930	0.0001	0.0125	0.1473

Table 7: Effect of s and S: fix $\eta = 2, \tau = 3, \alpha = 0.7, \beta = 0.8$

We notice that from Table 7 and Table 8 the measures, $E_I^{(f)}, E_F^{(f)}, E_P^{(f)}$ and $E_R^{(f)}$, increase as s increases for both positive and negative correlation. As s increases, $E_N^{(f)}$ increases for both arrival processes. However, much higher rate $E_N^{(f)}$ in P_{MAP}. Table 7 shows that $E_L^{(f)}$ and $E_{imp}^{(f)}$

	$S = 15$ with N_{MAP}				
s	$E_N^{(f)}$	$E_I^{(f)}$	$E_P^{(f)}$	$E_F^{(f)}$	$E_R^{(f)}$
6	0.5979	10.6006	1.1791	0.0812	0.1238
7	0.6067	11.1354	1.1902	0.0921	0.1397
8	0.6159	11.6112	1.2020	0.1035	0.1591
9	0.6273	12.1421	1.2158	0.1171	0.1858
10	0.6393	12.5997	1.2306	0.1314	0.2204
	$s = 10$ with N_{MAP}				
S	$E_N^{(f)}$	$E_I^{(f)}$	$E_P^{(f)}$	$E_F^{(f)}$	$E_R^{(f)}$
15	0.6393	12.5997	1.2306	0.1314	0.2204
16	0.6273	13.1421	1.2158	0.1171	0.1858
17	0.6159	13.6112	1.2020	0.1035	0.1591
18	0.6067	14.1353	1.1903	0.0921	0.1397
19	0.5979	14.6005	1.1792	0.0812	0.1238

Table 8: Effect of s and S: fix $\eta = 2, \tau = 3, \alpha = 0.7, \beta = 0.8$

decrease as s increases, as is to be expected. In the negative correlation case, the rate $E_L^{(f)}$ and $E_{imp}^{(f)}$ is zero so not included in Table 8 for both cases as increase of s and S.

Table 7 and Table 8 shows that, as S increases, $E_N^{(f)}$ increases for both arrival processes. However, much higher rate $E_N^{(f)}$ in positive correlation. We notice that the measures, $E_P^{(f)}, E_F^{(f)}$, and $E_R^{(f)}$, decrease as S increases for both positive and negative correlation. As S is increased, $E_I^{(f)}$ increases for both arrival processes in similar way.

Table 9 shows that $p_0^{(f)}, E_I^{(f)}$ increase as β increases for both arrivals, as expected line. As β increases, $E_N^{(f)}, E_F^{(f)}, E_{imp}^{(f)}, E_R^{(f)}, E_P^{(f)}$ decrease. However, much higher decrease of $E_N^{(f)}$ in negative correlation.

7.3. Revenue Function

Based on performance measures define the revenue (profit) function of finite case as follows:

$$F^f(\beta, s, S) = C_P E_P^{(f)} + C_F E_F(f) - [C_I E_I(f) + C_N E_N(f) + C_L E_L(f) + [\mathbf{F} + (S-s)C_R] E_R(f)$$
$$+ C_{imp} E_{imp}(f)]$$

where $C_P, C_F, C_I, C_N, C_R, C_{imp}, \mathbf{F}$ are given in Sec. 6.1.

	P_{MAP}							
β	$p_0^{(f)}$	$E_N^{(f)}$	$E_I^{(f)}$	$E_P^{(f)}$	$E_F^{(f)}$	$E_L^{(f)}$	$E_{imp}^{(f)}$	$E_R^{(f)}$
0.1	0.0003	190.5045	17.0535	2.9990	2.6326	0.0001	0.0669	0.1842
0.2	0.0040	166.3625	17.0574	2.9879	2.2579	0.0001	0.0582	0.1835
0.3	0.0730	105.7098	17.1225	2.7810	1.7621	0.0001	0.0368	0.1712
0.4	0.2027	70.0076	17.2294	2.3918	1.2242	0.0001	0.0242	0.1480
0.5	0.3150	50.7281	17.3057	2.0551	0.8019	0.0001	0.0174	0.1278
0.6	0.4028	38.3440	17.3537	1.7917	0.4813	0.0001	0.0130	0.1118
0.7	0.4723	29.1902	17.3840	1.5832	0.2330	0.0000	0.0098	0.0992
0.8	0.5289	21.7482	17.4039	1.4132	0.0364	0.0000	0.0072	0.0888
0.9	0.5764	15.3958	17.4173	1.2708	0.0115	0.0000	0.0050	0.0801
1	0.6167	10.0121	17.4263	1.1499	0.0024	0.0000	0.0031	0.0727
	N_{MAP}							
β	$p_0^{(f)}$	$E_N^{(f)}$	$E_I^{(f)}$	$E_P^{(f)}$	$E_F^{(f)}$	$E_L^{(f)}$	$E_{imp}^{(f)}$	$E_R^{(f)}$
0.1	0.0003	192.2960	17.0535	2.9991	2.6327	0.0001	0.0675	0.1842
0.2	0.0004	183.9908	17.0536	2.9989	2.2662	0.0001	0.0644	0.1842
0.3	0.0147	46.0626	17.0694	2.9560	1.8727	0.0001	0.0157	0.1817
0.4	0.2492	2.5583	17.3045	2.2523	1.1538	0.0000	0.0004	0.1402
0.5	0.3985	1.3257	17.4257	1.8045	0.7061	0.0000	0.0001	0.1136
0.6	0.4982	0.9059	17.4852	1.5053	0.4073	0.0000	0.0000	0.0957
0.7	0.5696	0.6918	17.5120	1.2913	0.1938	0.0000	0.0000	0.0828
0.8	0.6231	0.5608	17.5086	1.1307	0.0334	0.0000	0.0000	0.0730
0.9	0.6649	0.4715	17.4253	1.0052	0.0092	0.0000	0.0000	0.0650
1	0.6983	0.4070	17.4069	0.9050	0.0001	0.0000	0.0002	0.0587

Table 9: Effect of β: fix $s = 10, S = 25, \eta = 2, \tau = 3, \alpha = 0.7, \beta = 0.8$

For different values of S and s the revenue function is calculated and presented in Table 10. This table shows that the revenue of the system first increases and then decreases in both positive ana negative correlation.

Table 11 represents the revenue to the system as a function of β for MAP with positive and negative correlations. In both cases the revenue function behaves as a concave function and maximum revenue at $\beta = 0.3$ (see figure 4).

8. Conclusion

In this paper, we develop single server queuing-inventory model MAP/PH/1 with instantaneous feedback scheme. Both types of models with infinite and finite common buffer for waiting for arriving primary and feedback customers are examined. It is assumed that the service of both types' customers (primary

	P_{MAP}				N_{MAP}		
S	$s=10$	$s=11$	$s=12$	S	$s=10$	$s=11$	$s=12$
17	103.6453	100.8089	96.9543	15	95.5882	90.8642	84.1191
18	104.1319	101.881	98.9611	16	96.4661	93.5884	88.8643
19	**104.2067**	102.3442	100.0168	17	**96.8238**	94.4663	91.5884
20	103.9945	**102.3991**	100.4661	18	96.4702	**94.8239**	92.4663
21	103.5761	102.1698	**100.5092**	19	95.9797	94.4702	**92.8239**
22	103.0059	101.7366	100.2699	20	95.1274	93.9797	92.4702
23	102.3218	101.1536	99.8282	21	94.2725	93.1274	91.9797
	P_{MAP}				N_{MAP}		
s	$S=15$	$S=20$	$S=25$	s	$S=15$	$S=20$	$S=25$
3	108.6951	105.9626	101.9848	1	106.8733	101.5580	95.7669
4	109.3045	106.8088	102.7978	2	**107.5024**	**101.7656**	**95.8154**
5	**109.4870**	107.2855	103.2268	3	106.9592	101.3194	95.0968
6	109.1609	**107.3539**	**103.2674**	4	106.1966	100.5575	94.5176
7	108.2281	107.0166	102.9663	5	105.0994	99.8438	93.6620
8	106.5892	106.3153	102.3971	6	103.9698	98.9601	93.0009
9	104.1020	105.2984	101.6290	7	102.4663	98.1621	92.1482

Table 10: Effect of s and S on $F^f(\beta, s, S)$

β	P_{MAP}	N_{MAP}
0.1	122.6979	120.9142
0.2	143.7435	127.3801
0.3	**178.6965**	**258.6448**
0.4	168.0016	219.2900
0.5	147.4019	167.9145
0.6	128.6784	133.2594
0.7	113.2592	108.4459
0.8	100.7125	89.8568
0.9	90.3736	75.5670
1	81.6212	64.0211

Table 11: Effect of β on $F^f(\beta, s, S)$

or feedback) requires a unite size of inventory. The admission of an arriving p-customer is controlled by the Bernoulli trial when the inventory level at that moment is zero: according to the Bernoulli trial, the p-customer either joins the queue or leaves the system. After completing service, the customer can be feedback, i.e., decide to either join the common queue or leave the system in accordance with Bernoulli's trial, the parameters of which depend on the current

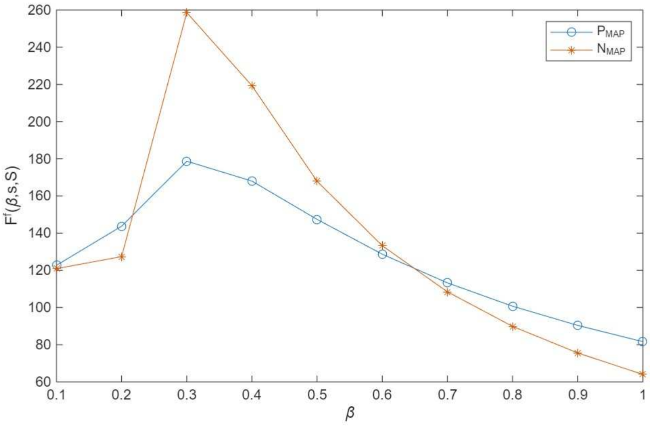


Figure 4: Effect of β on $F^f(\beta, s, S)$

inventory level. Both kinds of customers can be feedback, i.e., multiple feedback is allowed. Customers in queue become impatient when inventory levels drop to zero: in such cases, customers independently leave the system after waiting some exponentially distributed period of time. In order to be specific, an (s, S)-type inventory replenishment policy is used and the lead time is assumed to have exponential distribution. Generalizing the obtained results to other replenishment policies is straightforward.

The stability condition of the model with an infinite buffer for waiting for different types of customers is established. Key performance measures of finite and infinite buffer systems are determined using the matrix-geometric method. Long-term total expected revenue rate is calculated and maximized. The influence of various model parameters (warehouse capacity, reorder point, Bernoulli trial parameters, load parameters, etc.) on performance measures as well as on the revenue rate is numerically illustrated.

Further research directions could include generalizing the obtained results for models with both types of feedback (instantaneous and delayed), as well as studying a similar model with state-dependent server vacation.

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Agassi Melikov
Baku Engineering University, Azerbaijan
E-mail: agassi.melikov@gmail.com & amelikov@beu.edu.az

Dhanya Shajin
Department of Mathematics, T K Madhava Memorial College, Nangiarkulangara, Alappuzha, Kerala-690513, India.
E-mail: dhanya.shajin@gmail.com

Sevinj Aliyeva
Baku Engineering University, Azerbaijan
E-mail: s@aliyeva.info

Janos Sztrik
Department of Informatics and Networks, Faculty of Informatics, University of Debrecen, 4032 Debrecen, Hungary.
E-mail: sztrik.janos@inf.unideb.hu

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