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Description of the Domain of Definition of the Fractional Power $L^{\frac{1}{2}}$ of a Second-order Differential Operator

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Abstract. In this paper, we study the operator L generated in the space $L_p(0,1)$ by a second-order differential operator with integral boundary conditions: $U_v(y) = \int_0^1 \varphi_v(x)y(x)\,dx$, $v = 1, 2$. Such an operator is not densely defined in any space $L_p(0,1)$. Therefore, the operator is considered not on the whole $L_p(0,1)$, but in its subspace $L_{p,U}(0,1) = \{y(x) \in L_p(0,1) : U_v(y), v = 1,2\}, 1 < p < \infty$, which has codimension 2. Under additional conditions on the functions $\varphi_v(x), v = 1,2$, a constructive description of the domain of definition of the operator $L^{\frac{1}{2}}$ is given. The results on uniform and absolute convergence of biorthogonal expansions in eigen- and associated functions are formulated.

Key Words and Phrases: fractional power, positive operator, boundary condition, domain of operator, differential operator.

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1. Introduction

Consider the linear differential expression

$$l(y) = -y^{''} + q(x)y, x \in (0,1) \tag{1}$$

and boundary conditions $U_v(y) = 0, v = 1,2$, where $q(x)$- is a complex-valued function summable on $[0,1]$ and $U_1(y)$ and $U_2(y)$-are the corresponding boundary forms. Differential expression (1) and boundary conditions $U_v(y) = 0, v = 1,2$, generate a differential operator L with a domain of definition $D(L)$ in some functional space X. We are interested in some spectral properties, including the behavior of the eigenvalues and the basis properties of the eigenfunctions, as well

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as the domains of fractional powers of this operator Such a problem in the case of regular boundary conditions has been studied quite well (see [1, 2] and the bibliography there). The case of irregular, as well as more general regular boundary conditions, when the boundary conditions contain some integrals of the function $y(x)$ and its derivatives, was considered in [3, 8]. In these works, the spectral properties of the corresponding operator were studied in the space $L_2(0,1)$. The spectral properties of differential operators with regular two-point boundary conditions in $L_p(0,1)$. spaces were studied in [9, 11]. In [12, 19], direct and inverse spectral problems for differential operators with multipoint boundary conditions, as well as with discontinuity conditions in various function spaces, were studied. We also note the class of degenerate boundary conditions for which the spectrum of the corresponding operator is either empty or coincides with the entire complex plane (see [20] and the bibliography therein). However, as a rule, boundary forms generated an unbounded functional in the space under consideration, and in this case the operator has a dense domain of definition, which made it possible to construct a conjugate operator or assume the regularity of boundary conditions [1, 2, 4]. Here we will consider integral boundary conditions

$$U_v(y) = \int_0^1 \varphi_v(x) y(x)\, dx,\ v = 1, 2 \tag{2}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are given linearly independent functions belonging to the space $L_q(0,1), \frac{1}{p} + \frac{1}{q} = 1$. These conditions are not regular in the sense of Birkhoff [1], and there is no corresponding conjugate operator for them. Such conditions were used for other purposes in [6, 7]. In [21] problem (1), (2) was investigated under conditions of sufficient smoothness of the functions $q(x)$ and $\varphi_v(x)$, the asymptotic behavior of the eigenvalues and eigenfunctions was studied, and in [22] a theorem on the Riesz basis property of a system of eigenfunctions in a certain subspace $L_2(0,1)$ was proven. Note that differential equations with nonlocal conditions of integral form have interesting applications in mechanics [23] and in the theory of diffusion processes [24].

The theory of fractional powers of positive operators is closely related to questions of convergence of spectral decompositions. In [25], an application of fractional powers of self-adjoint operators to questions of uniform convergence of series in eigenfunctions of differential operators was given. More profound results in this direction were obtained in [26] for elliptic operators. In [28, 29], fractional powers of operators close to self-adjoint operators were used to estimate the rate of convergence of series in the eigen-functions of elliptic and pseudo-differential operators.

In the theory of boundary value problems, one of the important questions is the description of the domains of definition of fractional powers of positive operators. This question is the subject of the works [29-32], where positive operators generated by elliptic boundary value problems are studied and the domains of definition of fractional powers of such operators are described. In the papers [33, 34], the domains of definition of fractional powers of ordinary differential operators generated by regular boundary conditions in the space $L_p(0,1), 1 < p < \infty$, are described. In [10-12], generalizations of these results are given to the case of quasi-differential and differential operators with regular multipoint and integral boundary conditions. In [35, 36] the basis properties of the eigen- and associated functions of problem (1),(2) in a certain subspace $L_{p,U}(0,1) \subset L_p(0,1)$ of finite codimension were studied. In the present paper, we will give a constructive description of the domain of definition of the operator $L^{\frac{1}{2}}$.

2. Some auxiliary concepts and facts

Let's give some concepts and facts that we will need later

Definition 1. *[38] Let X-be a Banach space and A-be a closed linear operator with dense domain $D(A) \subset X$ and with values also in X. The operator A is called positive if the interval $(-\infty, 0]$ belongs to the resolvent set and there exists a number $C > 0$ such that*

$$\|(A + tI)^{-1}\| \leq \frac{C}{(1+t)}, t \geq 0 \tag{3}$$

Definition 2. *Let X-be a Banach space and let L- be a closed linear operator with dense domain $D(A) \subset X$ and with values also in X. A ray $l = \{\lambda; arg\lambda = \varphi\}$ is called a ray of minimal growth of the resolvent of the operator L, if the resolvent $R(\lambda) = (L - \lambda I)^{-1}$ exists on this ray far enough from the origin and satisfies the inequality*

$$\|R(\lambda)\| \leq \frac{C}{|\lambda|}.$$

From Definition 2 it follows that there exist numbers ε and h such that the operator $A = \varepsilon L + hI$ is positive.

Let A-be a positive operator. Then it is easy to show [39, p.135] that estimate (3) is also satisfied in some neighborhood of the interval $(-\infty, 0]$. Let Γ denote the contour that goes around the interval $(-\infty, 0]$ and is located in this neighborhood. For $-\infty < Rez < 0$ the complex powers of the operator A are defined by the formula [39, p.135]

$$A^z = \frac{1}{2\pi i} \int_\Gamma \lambda^z R(\lambda) d\lambda. \tag{4}$$

Complex powers of A^z, $-\infty < -Rez < 0$, are defined as operators inverse to powers of A^{-z}.

We present another theorem, which we will use to obtain estimates for the norms of the operators A^z.

Theorem 1. *(M.Riesz) [9]. Let $f \in L_p(0,1), 1 < p < \infty$. Then the integral*

$$g(x) = \int_0^1 \frac{f(t)}{x+t} dt$$

exists almost everywhere in $[0,1]$. Moreover, there exists a constant $C > 0$ such that the inequality $\|g\|_p \leq C\|f\|_p$ holds.

Let us introduce in the space $L_p(0,1), 1 < p < \infty$, a differential operator L, corresponding to the differential expression $l(y)$ with the domain of definition $D(L) = \{y(x) \in W_p^2(0,1), l(y) \in L_p(0,1); U_v(y) = 0, v = 1,2\}$ and consider the problem of the eigenvalues of this operator: $Ly = \lambda y$.

Let's put $\lambda = \rho^2$. Let us denote $S_\gamma = \{\rho : \frac{\gamma\pi}{2} \leq arg\rho \leq \frac{(\gamma+1)\pi}{2}\}, \gamma = 0,1,2,3$. In each region S_γ, equation (1) has a fundamental system of solutions with asymptotics [1, p. 58]

$$y_1(x,\rho) = e^{\rho\omega_1 x}(1 + \Upsilon_1(x,\rho)), y_2(x,\rho) = e^{\rho\omega_2 x}(1 + \Upsilon_2(x,\rho)), \tag{5}$$

where the numbers ω_1 and ω_2 are different square roots of (-1) (i.e. $\pm i$), numbered so that $Re(\rho\omega_1) \leq 0 \leq Re(\rho\omega_2)$ is satisfied for $\rho \in S_\gamma$, and the functions $\Upsilon_i(x,\rho)$ are continuous even for sufficiently large values of $|\rho|$ the estimate $|\Upsilon_i(x,\rho)| \leq \frac{c_i}{|\rho|}, i = 1,2$ is satisfied, uniformly in $x \in [0,1]$.

In what follows, with respect to functions $\varphi_\nu(x), \nu = 1,2$, we will assume that the following conditions are met:

A) $q(x) \in L_1(0,1); \exists\alpha \in (0,1) : \varphi_v(x) \in L_1(0,1) \cap W_1^1(0,\alpha) \cap W_1^1(1-\alpha,1), v = 1,2$;

B) $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, where $\alpha_v = \varphi_v(0)$, $\beta_v = \varphi_v(1)$.

Then in some strip $|\operatorname{Im}\rho| \leq \tau$, for some $\tau > 0$, the following relations are satisfied:

$$\int_0^1 \varphi_v(x) e^{i\rho x} dx = \frac{1}{i\rho}\left(\beta_v e^{i\rho} - \alpha_v\right) + o\left(\frac{1}{\rho}\right), \tag{6}$$

$$\int_0^1 \varphi_v(x) e^{-i\rho x} dx = \frac{1}{-i\rho}\left(\beta_v e^{-i\rho} - \alpha_v\right) + o\left(\frac{1}{\rho}\right),$$

These relations are obtained using integration by parts and from the Riemann-Lebesgue theorem. In addition, from (5) and (6) it follows that under the same assumptions the relations are also satisfied:

$$\left.\begin{aligned} U_\nu(y_1) = \int_0^1 \varphi_\nu(x)\, y_1(x,\rho)\, dx = \tfrac{1}{i\rho}\left(\beta_\nu e^{i\rho} - \alpha_\nu\right) + \tfrac{r_{\nu 1}(\rho)}{\rho} \\ U_\nu(y_2) = \int_0^1 \varphi_\nu(x)\, y_2(x,\rho)\, dx = \tfrac{1}{-i\rho}\left(\beta_\nu e^{-i\rho} - \alpha_\nu\right) + \tfrac{r_{\nu 2}(\rho)}{\rho} \end{aligned}\right\} \tag{7}$$

where for functions $r_{vi}(\rho)$ for large values of $|\rho|$ and $|\operatorname{Im}\rho| \leq \tau$ the estimate $r_{vi}(\rho) = o(1)$ is satisfied.

The eigenvalues of the operator L are the numbers $\lambda_n = \rho_n^2$, where ρ_n are the zeros of the characteristic determinant

$$\Delta(\rho) = \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{vmatrix}.$$

The following theorem is true regarding the function $\Delta(\rho)$ [37].

Theorem 2. [37] *Let conditions A), B) be met. Then for the characteristic determinant $\Delta(\rho)$ of the spectral problem (1), (2) the following are valid:*

i) *any number $\delta > 0$ corresponds to a constant $m_\delta > 0$, depending on the function $\Delta(\rho)$, such that on the set obtained from the complex ρ-plane by throwing out the δ-neighborhoods of the zeros of $\Delta(\rho)$ the inequality holds*

$$|\Delta(\rho)| \geq m_\delta \frac{1}{|\rho|^2} e^{\operatorname{Re}(\rho\omega_\delta)},$$

ii) *the zeros of the function $\Delta(\rho)$ are asymptotically simple and separated;*

iii) *the function $\Delta(\rho)$ has two series of roots: the first series has an asymptotic*

$$\rho_n = \pi n + o(1),$$

and the second series ρ'_n is defined by the equality $\rho'_n = -\rho_n$.

The operator L constructed above does not have a dense domain of definition in the space $L_p(0,1)$ and therefore the eigenfunctions of the operator L cannot be complete in this space. To eliminate this drawback, consider the operator L not on the whole space $L_p(0,1)$, but in its closed subspace

$$L_{p,U}(0,1) = \{f(x) \in L_p(0,1) : U_\nu(f) = 0, \nu = 1, 2\}.$$

It is obvious that $codim L_{p,U} = 2$. Similarly we define the space

$$W_{p,U}^k(0,1) = \{f(x) \in W_p^k(0,1) : U_\nu(f) = 0, \nu = 1, 2\}.$$

Let us define the operator L in the space $L_{p,U}(0,1)$ as follows:

$$D(L) = \left\{y \in W^2_{p,U}(0,1) : \ell(y) \in L_{p,U}(0,1)\right\} \quad \text{and for } y \in D(L) : Ly = \ell(y).$$

The operator L thus defined has an everywhere dense domain of definition in $L_{p,U}(0,1)$ [40, Lemma 2.2]. To study the spectral properties of the operator L in the space $L_{p,U}(0,1)$, we construct and estimate the resolvent of the operator L. It is known (see [1, p. 47]) that the Green's function of the operator $L-\rho^2 I$ has the form

$$G(x,\xi,\rho) = \frac{1}{\Delta(\rho)} \begin{vmatrix} g(x,\xi,\rho) & y_1(x,\rho) & y_2(x,\rho) \\ U_1(g) & U_1(y_1) & U_1(y_2) \\ U_2(g) & U_2(y_1) & U_2(y_2) \end{vmatrix}, \tag{8}$$

where

$$g(x,\xi,\rho) = \begin{cases} -y_1(x,\rho) z_1(\xi,\rho), & x \geq \xi, \\ y_2(x,\rho) z_2(\xi,\rho), & x < \xi, \end{cases}$$

$$z_1(\xi,\rho) = \frac{y_2(\xi,\rho)}{W(\rho)}, \quad z_2(\xi,\rho) = \frac{y_1(\xi,\rho)}{W(\rho)}, \quad W(\rho) = \begin{vmatrix} y_1(\xi,\rho) & y_2(\xi,\rho) \\ y_1'(\xi,\rho) & y_2'(\xi,\rho) \end{vmatrix}$$

Consider in the complex ρ-plane the region $\Omega_\delta = \bigcap_{n=1}^{\infty} \{\rho : |\rho - \rho_n| \geq \delta\}$, where $\{\rho_k\}$ is the set of zeros of the function $\Delta(\rho)$. Let K_δ denote the region of the complex λ-plane, which is the image of Ω_δ under the mapping $\lambda = \rho^2$.

In work [37] it is shown that for the Green's function $G(x,\xi,\rho)$, the following representation is true:

$$G(x,\xi,\rho) = \frac{1}{\rho} P_0(\rho,x,\xi) + \frac{1}{\rho} \sum_{i,k=1}^{2} P_{i,k}(\rho,x,\xi) \tag{9}$$

where

$$P_0(\rho,x,\xi) = A_{00}(\rho,x,\xi) E_0(\rho,x,\xi), \quad P_{ik}(\rho,x,\xi) = A_{ik}(\rho,x,\xi) E_i(\rho,x) E_k(\rho,\xi), \tag{10}$$

$$A_{ik}(\rho,x,\xi) = a_{ik}(\rho)(1+u_i(\rho,x))(1+\nu_k(\rho,\xi)), \quad i,k = 0,1,2, \tag{11}$$

$$E_0(x,\xi,\rho) = \begin{cases} e^{i\rho(x-\xi)}, & x > \xi, \\ e^{i\rho(\xi - x)}, & x < \xi, \end{cases} \quad E_1(\rho,x) = e^{i\rho x}, \quad E_2(\rho,x) = e^{i\rho(1-x)} \tag{12}$$

if $\rho \in S_0 \cap \Omega_\delta$ and

$$E_0(x,\xi,\rho) = \begin{cases} e^{-i\rho(x-\xi)}, & x > \xi, \\ e^{-i\rho(\xi-x)}, & x < \xi, \end{cases} \quad E_1(\rho,x) = e^{-i\rho x}, \quad E_2(\rho,x) = e^{-i\rho(1-x)} \tag{13}$$

if $\rho \in S_3 \cap \Omega_\delta$. In addition, the following relations are fulfilled:

$$|a_{ik}(\rho)| \le c, \quad i,k = 0,1,2; \quad \rho \in (S_0 \cup S_3) \cap \Omega_\delta; \tag{14}$$

$$u_i(\rho,x) \to 0, \ \nu_k(\rho,\xi) \to 0 \text{ as } \rho \to 0 \text{ uniformly in } x,\xi \in [0,1]. \tag{15}$$

The following theorem is also proved in [37].

Theorem 3 *[37]. Let ρ take values on a ray on which $Re(\pm i\rho) \neq 0$, and let the operator L be generated by the differential expression (1) and boundary conditions (2), where conditions A), B) are satisfied. Then, for the resolvent of the operator L, for sufficiently large values of $|\rho|$, the following estimate holds:*

$$\|R(\rho^2)\|_{L_{p,U} \to L_{p,U}} \le \frac{c}{|\rho|^2} \tag{16}$$

It follows from this theorem that, in addition to the positive real semi-axis, all rays on the λ-plane emanating from the origin are rays of minimal growth of the resolvent of the operator L. Moreover, for sufficiently large $h > 0$, the operator $A = L + hI$ is positive.

Remark 1. *Let $f \in L_p(a,b)$ and $g = R(\rho^2)f$. Then, on the rays $Re(\pm i\rho) \neq 0$, the estimate*

$$\|g'\|_p \le \frac{C}{|\rho|}\|f\|_p. \tag{17}$$

This follows from the fact that for the function $\frac{1}{\rho}\frac{\partial}{\partial x}G(x,\xi,\rho)$, an asymptotic representation similar to (9) holds.

In the next section, we will give a constructive description of $D\left(A^{\frac{1}{2}}\right)$ under additional conditions on the functions $\varphi_\nu(x)$, $\nu = 1,2$. But before that, we will prove one more theorem in the general case.

Theorem 4 *Let the operator L be generated by the differential expression (1) and the boundary conditions (2), and $A = L + hI$. Let $f \in L_{p,U}(0,1)$ and set $g = A^{-1/2}f$. Then $g \in W^1_{p,U}(0,1)$ and the following estimate holds:*

$$\|g\|_p \le C\|f\|_p \tag{18}$$

where $C > 0$-is a constant independent of f and g. In other words, the operator $A^{-\frac{1}{2}}$ acts boundedly from $L_{p,U}(0,1)$ to $W^1_{p,U}(0,1)$.

Proof. Setting $z=-\frac{1}{2}$ in formula (4), we obtain

$$A^{-\frac{1}{2}}f=\frac{1}{\pi}\sin\frac{\pi}{2}\left\{\int_{N^2}^{\infty}\left[(N^2-t)^{-\frac{1}{2}}-t^{-\frac{1}{2}}\right](L+tI)^{-1}fdt+\right.$$

$$\left.+\int_{N^2}^{\infty}t^{-\frac{1}{2}}(L+tI)^{-1}fdt\right\}\stackrel{df}{=}\frac{1}{\pi}\left\{g_1+g_2\right\}. \qquad (19)$$

To prove the existence of the derivative of the function $g_1(x)$ for $x\in(0,1)$, we represent it in the form

$$g_1(x)=\int_{N^2}^{\infty}\left[(N^2-t)^{-\frac{1}{2}}-t^{-\frac{1}{2}}\right]\int_0^1 G(x,\xi,-t)f(\xi)d\xi dt=$$

$$=2\int_N^{\infty}\rho\left[(N^2-\rho^2)^{-\frac{1}{2}}-\rho^{-1}\right]\int_0^1 G(x,\xi,-\rho^2)f(\xi)d\xi d\rho. \qquad (20)$$

We also denote

$$\phi_{1j}(x)=2\int_N^{\infty}\rho\left[(N^2-\rho^2)^{-\frac{1}{2}}-\rho^{-1}\right]\times$$

$$\times\int_0^1 G_x^{(j)}(x,\xi,-\rho^2)f(\xi)d\xi d\rho,\quad j=0,1. \qquad (21)$$

Taking into account the relation

$$\left[(N^2-\rho^2)^{-\frac{1}{2}}-\rho^{-1}\right]=O\left(\frac{1}{\rho^3}\right)$$

and inequalities (16),(17), we obtain that the integral over ρ in (21) converges absolutely for $j=0$, and for $j=1$, it converges in $L_p(0,1)$. However, since (21) is obtained from (20) by formal differentiation, it follows that $\phi_{10}\in W_p^1(0,1)$ and $g_1'(x)=\phi_{11}(x), x\in(0,1)$, hence $g_1\in W_p^1(0,1)$.

Now let's consider the function $g_2(x)$. Taking into account (9), we represent it as follows:

$$g_2(x)=\frac{1}{\pi}\int_N^{\infty}\rho^{-1}\int_0^1 P_0(x,\xi,\rho)f(\xi)d\xi d\rho+$$

$$+\sum_{i,k=1}^{2}\int_N^{\infty}\rho^{-1}\int_0^1 P_{ik}(x,\xi,\rho)f(\xi)d\xi d\rho\stackrel{df}{=}\frac{1}{\pi}\left\{g_{2,0}(x)+g_{2,1}(x)\right\}. \qquad (22)$$

Let us also denote for $x\in(0,1)$,

$$\phi_{2,j}(x)=\int_N^{\infty}\rho^{-1}\int_0^1\left\{2\rho G_x^{(j)}(x,\xi,-\rho^2)-P_{0,x}^{(j)}(x,\xi,\rho)\right\}f(\xi)d\xi d\rho=$$

$$= \lim_{\varepsilon\to+0}\int_N^\infty \rho^{-1}\int_\varepsilon^{1-\varepsilon}\left\{2\rho G_x^{(j)}(x,\xi,-\rho^2)-P_{0,x}^{(j)}(x,\xi,\rho)\right\}f(\xi)d\xi d\rho \overset{df}{=} \lim_{\varepsilon\to+0}\phi_{2,j}(x,\varepsilon).$$

Let $f(x)$ first be a continuous function. We will show that the integrals $\phi_{2,j}(x,\varepsilon)$ converge absolutely. Indeed,

$$|\phi_{2,j}(x,\varepsilon)| \le C\sum_{i,k=1}^{2}\int_N^\infty \rho^{j-1}\int_\varepsilon^{1-\varepsilon}|f(\xi)||e^{\rho\Omega_{ik}(x,\xi)}|d\xi d\rho \le$$

$$\le C\sum_{i,k=1}^{2}\int_\varepsilon^{1-\varepsilon}|f(\xi)|\int_N^\infty \rho^{j-1}e^{\mathrm{Re}\rho\Omega_{ik}(x,\xi)}d\rho d\xi \le$$

$$\le C_1\sum_{i,k=1}^{2}\int_\varepsilon^{1-\varepsilon}|f(\xi)|\frac{\rho^{j-1}e^{N\Omega_{ik}(x,\xi)}}{-\mathrm{Re}\rho\Omega_{ik}(x,\xi)}d\xi \le$$

$$\le C_2\int_\varepsilon^{1-\varepsilon}|f(\xi)|\left(\frac{1}{x+\xi}+\frac{1}{1+x-\xi}+\frac{1}{1-x+\xi}+\frac{1}{2-x-\xi}\right)d\xi. \qquad (23)$$

Since $x+1-\xi>0$ for $x\in[0,1], \xi\in[\varepsilon,1-\varepsilon]$, the integral on the right-hand side of (23) converges uniformly in $x\in[0,1]$. On the other hand, $\phi_{2,1}(x,\xi)$ is obtained from the integral

$$g_{2,1}(x,\varepsilon) = \int_N^\infty \rho^{-1}\int_\varepsilon^{1-\varepsilon} f(\xi)\left[2\rho G(x,\xi,-\rho^2)-P_0(x,\xi,\rho)\right]d\xi d\rho,$$

by differentiation. It follows that $g_{2,1}(x,\xi)\in W_p^1(0,1)$. Considering that

$$g_{2,1}(x) = \lim_{\varepsilon\to+0} g_{2,1}(x,\xi),$$

in the norm of $L_p(0,1)$, from the closeness of the differentiation operator $\frac{d}{dx}$, we obtain that $g_{2,1}(x,\varepsilon)\in W_p^1(0,1)$. Moreover, by Riesz's theorem 1, it follows from (23) that

$$\|g_{2,1}'\|_p \le C\|f\|_p.$$

To prove the theorem for the function $g_{2,0}(x)$, we consider the operator L_0, generated by the differential expression (1) and some strongly regular two-point boundary conditions. We set $A_0 = L_0 + hI$, where the choice of the number h ensures the positivity of the operator A_0. Let $g_0 = A_0^{-\frac{1}{2}}f, f\in L_p(0,1)$. In [10] (see also [11]) it was proved that estimate (18) holds for the function $g_0(x)$. On the other hand, in representation (9) for the Green's function of the operator $L-$

$\rho^2 I$ the function $P_0(x,\xi,\rho)$ coincides with the corresponding function $P_{00}(x,\xi,\rho)$ from the representation for the Green's function of the operator $L_0-\rho^2 I$, similar to (9). It follows that estimate (18) is also true for the function $g_{2,0}(x)$.

The theorem is proved. ◄

3. Constructive description of the domain of the operator $A^{\frac{1}{2}}$

Consider in the space $L_{p,U}(0,1)$ the operator L, generated by the differential expression $l(y)$ from (1) and the boundary conditions (2). Since the functions $\varphi_1(x)$ and $\varphi_2(x)$ from (2) are linearly independent, their linear span forms a two-dimensional subspace in $L_q(0,1)$ and, therefore, is complemented in $L_q(0,1)$. Let $L_{q,V}(0,1)$ be its complement. Then we can choose functions $\psi_1(x)$ and $\psi_2(x)$ from $L_p(0,1)$ such that $\langle\psi_\nu,\varphi_k\rangle, \nu,k=1,2$ and

$$V_\nu(y)=\int_0^1 \psi_\nu(x)z(x)dx=0, \nu=1,2, \tag{24}$$

for any $z(x)$ from $L_{q,V}(0,1)$. On the other hand, $W_q^2(0,1)$ is everywhere dense in $L_q(0,1)$, then $W_{q,V}^2(0,1)$ will also be everywhere dense in $L_{q,V}(0,1)$ (see [40, p. 28]. To construct the adjoint operator with respect to the functions $\varphi_1(x)$ and $\varphi_2(x)$, we additionally assume that they belong to $W_q^2(0,1)$ and, moreover, $\tilde{l}(\varphi_1),\tilde{l}(\varphi_1)\in L_q(0,1)$, where

$$\tilde{l}(z)=z''+\overline{q(x)}z.$$

Let $y\in D(L)$ and $z\in W_q^2(0,1)$. Then, integrating twice by parts, we obtain

$$\langle Ly,z\rangle=-y'(1)\overline{z(1)}+y'(0)\overline{z(0)}+y(1)\overline{z'(1)}-y(0)\overline{z'(0)}+\langle y,\tilde{l}(z)\rangle. \tag{25}$$

On the other hand, given that $l(y)\in W_{p,U}^2(0,1)$, the function $l(y)$ satisfies the boundary conditions (2). Then, integrating by parts twice, we have

$$\langle l(y),\overline{\varphi_1}\rangle=-y'(1)\varphi_1(1)+y'(0)\varphi_1(0)+y(1)\varphi_1'(1)-y(0)\varphi_1'(0)+\langle y,\tilde{l}(\varphi_1)\rangle=0,$$

$$\langle l(y),\overline{\varphi_2}\rangle=-y'(1)\varphi_2(1)+y'(0)\varphi_2(0)+y(1)\varphi_2'(1)-y(0)\varphi_2'(0)+\langle y,\tilde{l}(\varphi_2)\rangle=0,$$

According to the assumption $\Delta\overset{def}{=}\varphi_1(1)\varphi_2(0)-\varphi_2(1)\varphi_1(0)\neq 0$. Then, solving the resulting system of equations for the unknowns $y'(1)$ and $y'(0)$, we obtain

$$y'(1)=\frac{1}{\Delta}\left(\varphi_2(0)\varphi_1'(1)-\varphi_1(0)\varphi_2'(1)\right)y(1)+\frac{1}{\Delta}\left(\varphi_1(0)\varphi_2'(0)-\varphi_1'(0)\varphi_2(0)\right)y(0)+$$

$$+\frac{1}{\Delta}\langle y, \overline{\varphi_2(0)}\tilde{l}(\varphi_1) - \overline{\varphi_1(0)}\tilde{l}(\varphi_2)\rangle$$

$$y'(0) = \frac{1}{\Delta}\left(\varphi_2(1)\varphi_1'(1) - \varphi_1(1)\varphi_2'(1)\right) y(1) + \frac{1}{\Delta}\left(\varphi_2(1)\varphi_1'(0) - \varphi_2'(0)\varphi_1(1)\right) y(0) +$$

$$+\frac{1}{\Delta}\langle y, \overline{\varphi_2(1)}\tilde{l}(\varphi_1) - \overline{\varphi_1(1)}\tilde{l}(\varphi_2)\rangle.$$

Now substituting the obtained expressions into (25), we obtain

$$\langle L(y), z\rangle = y(1)\times$$

$$\times\left[\frac{1}{\Delta}\left(\varphi_2(0)\varphi_1'(1) - \varphi_1(0)\varphi_2'(1)\right)\overline{z(1)} + \frac{1}{\Delta}\left(\varphi_2(1)\varphi_1'(1) - \varphi_1(1)\varphi_2'(1)\right)\overline{z(0)} + \overline{z'(1)}\right] +$$

$$+y(0)\times$$

$$\times\left[\frac{1}{\Delta}\left(\varphi_1'(0)\varphi_2(0) - \varphi_1(0)\varphi_2'(0)\right)\overline{z(1)} + \frac{1}{\Delta}\left(\varphi_2(1)\varphi_1'(0) - \varphi_2'(0)\varphi_1(1)\right)\overline{z(0)} - \overline{z'(0)}\right] +$$

$$+\langle y, l(z)\rangle + \frac{1}{\Delta}\left(\varphi_2(0)\tilde{l}(\varphi_1) - \varphi_1(0)\tilde{l}(\varphi_2)\right) z(1) + \frac{1}{\Delta}\left(\varphi_2(1)\tilde{l}(\varphi_1) - \varphi_1(1)\tilde{l}(\varphi_2)\right) z(0)\rangle.$$

From the obtained relations it follows that to ensure the equality

$$\langle Ly, z\rangle = \langle y, L^* z\rangle,$$

the adjoint operator should be defined as follows:

$$D(L^*) = \left\{z \in W_q^2(0,1) : V_\nu(z) = 0,\ \nu = \overline{1,4};\ l^*(z) \in L_{q,V}(0,1)\right\} \tag{26}$$

and for $z \in D(L^*)$:

$$L^* z = l^*(z), \tag{27}$$

where

$$l^*(z) = -z'' + \overline{q(x)}z + \frac{1}{\Delta}\left(\varphi_2(0)\tilde{l}(\varphi_1) - \varphi_1(0)\tilde{l}(\varphi_2)\right) z(1) +$$

$$+\frac{1}{\Delta}\left(\varphi_2(1)\tilde{l}(\varphi_1) - \varphi_1(1)\tilde{l}(\varphi_2)\right) z(0), \quad x \in (0,1); \tag{28}$$

the boundary forms $V_1(z)$ and $V_2(z)$ are defined above (see (24)), and the forms $V_3(z)$ and $V_4(z)$ are defined by the equalities

$$V_3(z) = z'(1) + \frac{1}{\Delta}\left(\overline{\varphi_2(0)\varphi_1'(1)} - \overline{\varphi_1(0)\varphi_2'(1)}\right) z(1) +$$

$$+\frac{1}{\Delta}\left(\overline{\varphi_2(1)\varphi_1'(1)}-\overline{\varphi_1(1)\varphi_2'(1)}\right)z(0),$$

$$V_4(z)=z'(0)+\frac{1}{\Delta}\left(\overline{\varphi_1'(0)\varphi_2(0)}-\overline{\varphi_1(0)\varphi_2'(0)}\right)z(1)-$$

$$-\frac{1}{\Delta}\left(\overline{\varphi_2(1)\varphi_1'(0)}-\overline{\varphi_2'(0)\varphi_1(1)}\right)z(0).$$

Thus, the adjoint operator L^* is defined in the space $L_{q,V}(0,1)$ by the functional-differential expression (28) using equalities (26), (27).

The following lemma plays an important role in proving the main result.

Lemma 1. *Let $y\in D(L)$, $z\in W_q^1(0,1),\varphi_\nu\in W_q^2(0,1),\tilde{l}(\varphi_\nu)\in L_q(0,1),\nu=1,2,\frac{1}{p}+\frac{1}{q}=1$. Then the following estimate holds.*

$$|\langle Ly,z\rangle|\leq C\|y\|_{W_p^1(0,1)}\|z\|_{W_q^1(0,1)}. \tag{29}$$

Proof. Integrating by parts, we get

$$\langle Ly,z\rangle=\int_0^1(-y''(x)\overline{z'(x)}dx+q(x)y(x))\overline{z(x)}=-y'(1)\overline{z(1)}+y'(0)\overline{z(0)}+$$

$$+\int_0^1 y'(x)\overline{z'(x)}dx+\int_0^1 q(x)y(x)\overline{z(x)}dx\overset{df}{=}$$

$$\overset{df}{=}[y,z]+\langle y',z'\rangle+Q(y,z).$$

For $\langle y',z'\rangle$,estimate (29) follows from Hölder's inequality, and for $Q(y,z)$, it follows from the well-known estimate

$$\underset{x\in[0,1]}{max}|y^{(i)}(x)|\leq\|y\|_{W_p^{j+1}(0,1)},j=0,1. \tag{30}$$

Considering that for functions $y\in W_{p,U}^2(0,1)$ the function $l(y)$ also satisfies the boundary conditions (3), we obtain

$$U_\nu(l(y))=-\int_0^1 y''(x)\varphi_\nu(x)dx+\int_0^1 q(x)y(x)\varphi_\nu(x)dx=0.$$

From here, integrating by parts, we have

$$U_\nu(l(y))=y'(1)\varphi_\nu(1)-y'(0)\varphi_\nu(0)=\int_0^1 y'(x)\varphi_\nu'(x)dx+$$

$$+\int_0^1 q(x)y(x)\varphi_\nu(x)dx,\nu=1,2. \tag{31}$$

Solving the resulting system of linear equations for the unknowns $y'(1)$ and $y'(0)$, we obtain

$$y'(1) = \frac{1}{\Delta}(\varphi_2(0)b_1 - \varphi_1(0)b_2), y'(0) = \frac{1}{\Delta}(\varphi_1(1)b_2 - \varphi_2(1)b_1), \qquad (32)$$

where we denote

$$b_\nu = \int_0^1 y'(x)\varphi_\nu'(x)\,dx + \int_0^1 q(x)y(x)\varphi_\nu(x)\,dx, \quad \nu = 1, 2.$$

Then, substituting (32) into (31), we obtain the following expression for the form $[y, z]$:

$$[y, z] = -\frac{1}{\Delta}(\varphi_2(0)b_1 - \varphi_1(0)b_2)\,\overline{z(1)} + \frac{1}{\Delta}(\varphi_1(1)b_2 - \varphi_2(1)b_1)\,\overline{z(0)}.$$

Now estimate (29) for $[y, z]$ is obtained from inequalities (30) applied to the function $z(x)$ and from Hölder's inequality for b_ν. The lemma is proved. ◄

Remark 2. *From the construction of the adjoint operator, as well as from the form of the adjoint boundary conditions, it follows that an estimate similar to (29) is also valid for the adjoint operator.*

Theorem 5. *Let the operator L be generated by the differential expression (1) and the boundary conditions (2), and let $A = L + hI$. Furthermore, suppose that $\varphi_\nu \in W_q^2(0,1)$, $\tilde{l}(\varphi_\nu) \in L_q(0,1)$, $\nu = 1, 2$. Then the following relation holds:*

$$D\left(A^{\frac{1}{2}}\right) = W_{p,U}^1(0,1). \qquad (33)$$

Proof. First, we show that the inclusion $D(A^{\frac{1}{2}}) \subset W_{p,U}^1(0,1)$ holds. Let $f \in D(A^{\frac{1}{2}})$. Let E_n be the Riesz projections of the operator A, corresponding to the eigenvalues λ_n (see [37]). We expand the function f in the series

$$f \in \sum_{n=2}^{\infty} E_n f, \qquad (34)$$

which converges in $L_p(0,1)$ by Theorem 4 of [37]. On the other hand, Theorem 4 implies that series (34) can be differentiated term by term. Since the forms U_ν are continuous linear functionals in $W_{p,U}^1(0,1)$, by estimate (30), the functions $(E_n f)(x)$ satisfy all boundary conditions (2), acting by the functionals U_ν, $\nu = 1, 2$, on the series (34), we obtain $U_\nu(f) = 0$, $\nu = 1, 2$.

Now we prove the inverse inclusion. Let $g \in W^1_{p,U}(0,1)$, $z \in D(A^*)$ and set $\vartheta = (A^*)^{\frac{1}{2}} z$. Then $\vartheta \in D((A^*)^{\frac{1}{2}})$ and $(A^*)^{\frac{1}{2}}\vartheta = A^* z$. Applying Lemma 1 and then Theorem 4 to the adjoint operator, we obtain

$$\left|(g, (A^*)^{\frac{1}{2}}\vartheta)\right| = |(g, l^*(z))| \leq C\|g\|_{W^1_p(0,1)}\|z\|_{W^1_q(0,1)} =$$

$$= C\|g\|_{W^1_p(0,1)}\|(A^*)^{-\frac{1}{2}}\vartheta\|_{W^1_q(0,1)} \leq C_1\|g\|_{W^1_p(0,1)}\|\vartheta\|_{L_q(0,1)}.$$

Thus, $(g, (A^*)^{\frac{1}{2}}\vartheta)$- is a bounded functional of ϑ for any $g \in W^1_p(0,1)$. It follows that $W^1_{p,U}(0,1) \subset D(A^{\frac{1}{2}})$, and consequently, (33) holds.

The theorem is proved. ◄

Corollary 1. *The eigenfunctions and associated functions of the operator L generated by the expression $l(y)$ and the boundary conditions (2) form a basis in the space $W^1_{p,U}(0,1)$.*

Corollary 2. *The eigenfunctions and associated functions of the operator L generated by the expression $l(y)$ and the boundary conditions (2) form a Riesz basis in the space $W^1_{2,U}(0,1)$.*

Proof. Let $f \in W^2_{2,U}(0,1)$, then, according to [22], series (34) unconditionally converges in $L_2(0,1)$. However, the series

$$A^{\frac{1}{2}} f = \sum_{n=2}^{\infty} A^{\frac{1}{2}} E_n f$$

also converges unconditionally. It follows that series (34) converges unconditionally in the norm of the graph of the operator$A^{\frac{1}{2}}$ and, consequently, by Theorem 5, in the norm of the space $W^1_{2,U}(0,1)$. ◄

Corollary 3. *Let $f \in W^1_{p,U}(0,1)$. Then the series*

$$\sum_{n=2}^{\infty}(E_n f)(x) \tag{35}$$

converges uniformly to the function $f(x)$. Moreover, for $p = 2$, series (35) converges uniformly and absolutely.

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